Old and new in compensated compactness

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18 July 2022

A word on my work

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Today: Weak convergence effects.
Joint with André Guerra, Jan Kristensen, Matthew Schrecker.

The basic question

When do we have

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p \implies F(v_j) \overset{*}{\rightharpoonup} F(v) \text{ in } \mathscr{D}'?$$

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Ball, JL Lions, Morrey, Murat, Reshetnyak, Tartar:
Assume v_j have linear differential structure.
Find the restrictions on the nonlinearity F.

Two examples

Ball/Morrey/Reshetnyak:

$$Du_j \rightharpoonup Du \text{ in } \mathrm{L}^n(\mathbb{R}^n) \implies \det Du_j \overset{*}{\rightharpoonup} \det Du \text{ in } \mathcal{M}(\mathbb{R}^n).$$

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JL Lions–Murat–Tartar:

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Extraordinarily far-reaching: these come up in elasticity, fluids, electromagnetism, hyperbolic conservation laws, geometric analysis (harmonic maps between manifolds).

Murat's framework

For which nonlinearities F do we have

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We refer to such occurrences of weak sequential continuity as instances of *compensated compactness*. Could also include lower semi-continuity.

Very rough answer: for $\mathcal{A}$-quasiaffine (polynomials) F, i.e.,

$$F\left(\fint_{[0,1]^n} v(x)\,\mathrm{d}x\right) = \fint_{[0,1]^n} F(v(x))\,\mathrm{d}x \qquad (\mathcal{A}\text{-q})$$

for all $[0,1]^n$-*periodic* fields v with $\boxed{\mathcal{A}v = 0}$ (for terminology, compare with Dacorogna '81-'82, Fonseca–Müller '99).

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$s = q = p = 2$: answer is rigurous (Murat–Tartar '70s).

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All these examples have constant rank, i.e.,

$$\operatorname{rank} \mathcal{A}(\xi) \text{ is independent of } \xi \in \mathbb{R}^n \setminus \{0\},$$

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In what follows, $\mathcal{A}$ has **constant rank**.

Results

Set-up: Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $\mathcal{A}$ have constant rank, F $\mathcal{A}$-quasiaffine, s-homogeneous, $s \geq 2$, and $q \geq s$.
We examine:

$$\left.\begin{array}{ll} v_j \rightharpoonup v & \text{in } \mathrm{L}^q(\Omega) \\ \mathcal{A}v_j \to \mathcal{A}v & \text{in } \mathrm{W}^{-\ell,s}(\Omega) \end{array}\right\} \implies F(v_j) \overset{*}{\rightharpoonup} F(v) \text{ in } \mathscr{D}'(\Omega).$$

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- $q = s + c$ (Murat '81);
- $q = s + \varepsilon$ for any $\varepsilon > 0$ (Fonseca–Müller '99);
- $q = s$ (Guerra–R. '19).

Main difference: For us, $\{F(v_j)\}_j$ is not uniformly integrable.

Potentials for constant rank operators

Theorem (R. '18)

$\mathcal{A}$ has constant rank if and only if there exist a homogeneous $\mathcal{B}$ such that

$$\ker \mathcal{A}(\xi) = \operatorname{im} \mathcal{B}(\xi) \quad \text{for all } \xi \neq 0.$$

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$\mathcal{B}$ is explicitly computable by linear algebra techniques only. Also purely commutative algebra proof (Härkönen–Nicklasson–R. '21).

What about weak continuity?

Look at lower semi-continuity:

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The implication holds for $q > p$ if and only if F is $\mathcal{A}$-quasiconvex (i.e., "$\leqslant$" in ($\mathcal{A}$-q)).

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Bonus: Computation of all $\mathcal{A}$–quasiaffine functions is reduced to linear algebraic systems (uses Ball–Currie–Olver '81).

Surprising Hardy space regularity

Elasticity setting: Müller '91:

$$Du \in \mathrm{L}^n_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^{n\times n}) \implies \det Du \in \mathrm{L}\log\mathrm{L}_{\mathrm{loc}}(\mathbb{R}^n),$$

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Further developed in Coifman–PL Lions–Meyer–Semmes '93:

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CLMS ask whether "nonlinear weakly continuous quantities" and "nonlinear quantities that belong to $\mathscr{H}^1$" coincide.

Weak continuity vs. Hardy space regularity

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(WC) *weakly continuous quantity* iff

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Theorem (Guerra–R. '19)

If F is an s-homogeneous polynomial then (WC) $\iff$ (HQ).

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For $q < s$: distributional quantities, $\mathscr{H}^p$ bounds, $p = q/s < 1$.

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Below the integrability parameter:
CLMS: $u \in \mathrm{W}^{1,q}(\mathbb{R}^n, \mathbb{R}^n) \implies \det Du \in \mathscr{H}^{q/n}(\mathbb{R}^n)$.

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Refinement of the weak convergence of Jacobians.

Oscillation and concentration

Vitali's Convergence Theorem:
$v_j \to v$ in $\mathrm{L}^p \iff$ both

1. $v_j \to v$ in measure (no oscillation);
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Examples of $v_j \rightharpoonup 0$ in L^p when exactly one of these fails:

1. $v_j(x) = \sin(jx)$ (oscillation);
2. $v_j(x) = j^{1/p}\mathbf{1}_{(0,1/j)}$ (L^p-concentration).

Decomposition lemma for $\mathcal{A}$-free sequences

Origins: Kristensen '94, Fonseca–Müller–Pedregal '98, Fonseca–Müller '99.

Lemma (R. '19, Guerra–R.–Kristensen '20)

Let $1 < p < \infty$ *and*

$$v_j \rightharpoonup v \text{ in } \mathrm{L}^p, \quad \mathcal{A}v_j \to \mathcal{A}v \textit{ in } \mathrm{W}^{-\ell,p}.$$

Then there exist $u_j, \tilde{u}_j \in \mathrm{C}_c^\infty$ *such that* $\mathcal{B}u_j$, $\mathcal{B}\tilde{u}_j \rightharpoonup 0$ *in* L^p *s.t.*

$$\begin{aligned} & v_j = v + \mathcal{B}u_j + \mathcal{B}\tilde{u}_j \\ & \mathcal{B}u_j \textit{ is } p\textit{-uniformly integrable} \\ & \mathcal{B}\tilde{u}_j \to 0 \textit{ in measure.} \end{aligned}$$

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$$v_j = v + \mathcal{B}u_j + \mathcal{B}\tilde{u}_j$$

$\mathcal{B}u_j$ *is* p*-uniformly integrable*

$\mathcal{B}\tilde{u}_j \to 0$ *in measure.*

For $1 < p < \infty$ both oscillation and concentration effects of $\mathcal{A}$-free sequences have $\mathcal{A}$-free structure!

A word on another concentration phenomenon

For $p = 1$, look at $v_j \overset{*}{\rightharpoonup} v$ in $\mathcal{M}$, $\mathcal{A}v \to \mathcal{A}v$ in $\mathrm{W}^{-1,2}$.

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Thank you for the attention and the invitation!