Quantitative stability results in fluid mechanics and kinetic theory

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Introduction

Fluid mixing

Mixing can be thought of as a cascading process in which information travels to smaller and smaller spatial scales.

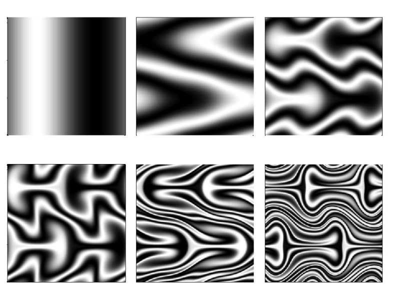

Figure 1: No diffusion (Doering et al.)

Understanding this fundamental process sheds light on:

- Relaxation towards stationary states and coherent structures
- Meta-stable behavior in ocean/atmospheric models
- The derivation of turbulence scaling laws (Kolmogorov, Batchelor)

Diffusive mixing

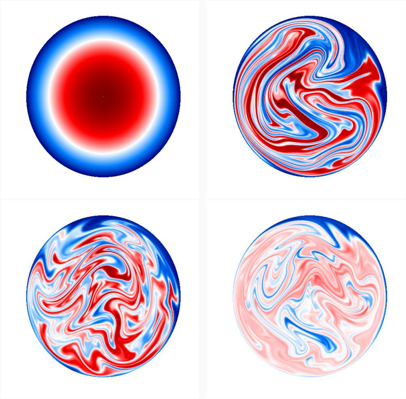

Figure 2: From J. Vanneste (Edinburgh)

The high and low concentrations of a scalar in a disc when stirred by a random flow.

- In fluid mechanic slang: "Mixing sends stuff to small scales, until dissipation kicks in and kills everything"
- Creation of filaments
- At the beginning: stirring is dominant
- At the end: diffusion is dominant

Fluid mechanics

The Navier-Stokes and Euler equations

In a 2d domain, consider

$$\begin{cases} \partial_t \boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} + \nabla P = \nu \Delta \boldsymbol{U}, \\ \nabla \cdot \boldsymbol{U} = 0. \end{cases}$$

- $\boldsymbol{U} = (U_1, U_2)$ is the velocity field of the fluid
- P is the scalar pressure

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    - $\nu = 0$: Inviscid fluid $\rightarrow$ Euler equations
    - $\nu > 0$: Viscous fluid $\rightarrow$ Navier-Stokes equations

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In vorticity formulation $\Omega = \nabla^{\perp} \cdot \boldsymbol{U} = -\partial_y U_1 + \partial_x U_2$:

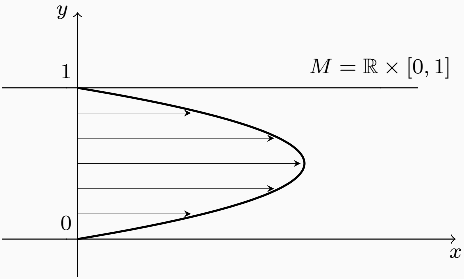
$$\begin{cases} \partial_t \Omega + \boldsymbol{U} \cdot \nabla \Omega = \nu \Delta \Omega, \\ \boldsymbol{U} = \nabla^{\perp} \Psi, \quad \Delta \Psi = \Omega. \end{cases}$$

Stationary flows

For a given equilibrium $\boldsymbol{U}_E$, write $\boldsymbol{U} = \boldsymbol{U}_E + \boldsymbol{u}$

$$\begin{cases} \partial_t \omega + \boldsymbol{U}_E \cdot \nabla \omega + \boldsymbol{u} \cdot \nabla \Omega_E = \nu \Delta \omega - \boldsymbol{u} \cdot \nabla \omega, \\ \boldsymbol{u} = \nabla^\perp \psi, \quad \Delta \psi = \omega. \end{cases}$$

Typical one: shear flows $\boldsymbol{U}_E = (u(y), 0)$



- Couette: $u(y) = y$, on $\mathbb{T} \times \mathbb{R}$
- Poiseuille: $u(y) = y^2$, on $\mathbb{T} \times \mathbb{R}$
- Kolmogorov: $u(y) = \sin y$, on $\mathbb{T}^2$

$$\partial_t \omega + u(y)\partial_x \omega - u''(y)\partial_x \psi = \nu \Delta \omega + \mathcal{N}$$

Asymptotic stability

Linearized stability

If $u(y) = y$:

$$\partial_t \omega + y\partial_x \omega = \nu \partial_{xx}\omega + \nu \partial_{yy}\omega$$

The solution is explicit: $\mathbb{T} \times \mathbb{R} \ni (x, y) \mapsto (k, \eta) \in \mathbb{Z} \times \mathbb{R}$

$$\hat{\omega}(k, \eta, t) = \hat{\omega}^{in}(k, \eta + kt) \exp\Big(-\nu \int_0^t \big(k^2 + (\eta + k\tau)^2\big) \mathrm{d}\tau\Big).$$

Linearized behavior near Couette

- If $\nu = 0$, info goes to high frequencies (when $k \neq 0$).
  Inviscid damping:

$$\|u_1(t) - \langle u_1 \rangle_x\|_{L^2} + \langle t \rangle \|u_2(t)\|_{L^2} \lesssim \langle t \rangle^{-1} \|\omega^{in}\|_{H^2}$$

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- If $\nu > 0$, then we have enhanced dissipation:

$$\|\omega(t) - \langle \omega(t) \rangle_x\|_{L^2} \leq \|\omega^{in} - \langle \omega^{in} \rangle_x\|_{L^2} \mathrm{e}^{-\frac{1}{12}\nu t^3}.$$

Nonlinear asymptotic stability

What happens at the nonlinear level?

For small perturbations, $\boldsymbol{u}(t,x,y) \to (u^{\infty}(y), 0)$ as $t \to \infty$

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For $\nu = 0$: $\|\omega^{in}\|_X \lesssim \varepsilon$ implies inviscid damping?

- Bedrossian, Masmoudi '13: if the perturbation is small in Gevrey-2^{-}, then inviscid damping holds.
- Deng, Masmoudi '18: Gevrey-2 is optimal.

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For $\nu > 0$: $\|\omega^{in}\|_X \lesssim \nu^\gamma$ implies enhanced dissipation?

- Bedrossian, Masmoudi, Vicol '14: $\gamma = 0$ if X is Gevrey-2^-.
- Masmoudi, Zhao '19: $\gamma = 1/3$ if $X = H^s$, $s > 40$.

Energy methods

Vector field method

Consider

$$\partial_t \omega + u(y)\partial_x \omega = 0 \qquad \Leftrightarrow \qquad \partial_t \omega_k + iku(y)\omega_k = 0.$$

The vector field $J_t = \partial_y + tu'(y)\partial_x$ commutes with the equation. Hence

$$\|J_t \omega\|_{L^2} = \|J_0 \omega^{in}\|_{L^2} = \|\partial_y \omega^{in}\|_{L^2}$$

If $|u'| \geq \delta > 0$, then

$$\begin{aligned} t\|\nabla \partial_x \psi_k\|_{L^2}^2 &= -t\langle \partial_x \psi_k, \partial_x \omega_k \rangle_{L^2} = -\langle \partial_x \psi_k, \frac{1}{u'} tu' \partial_x \omega_k \rangle_{L^2} \\ &= -\langle \partial_x \psi_k, \frac{1}{u'} J_t \omega_k \rangle_{L^2} + \langle \frac{1}{u'} \partial_x \psi_k, \partial_y \omega_k \rangle_{L^2} \\ &\lesssim_\delta \|\nabla \partial_x \psi_k\|_{L^2}(\|\omega_k\|_{L^2} + \|J_t \omega_k\|_{L^2}). \end{aligned}$$

Inviscid mixing for monotone shears

$$t\|\nabla \partial_x \psi\|_{L^2} \lesssim_\delta \|\omega\|_{L^2} + \|J_t \omega\|_{L^2} \lesssim_\delta \|\omega^{in}\|_{L^2} + \|\partial_y \omega^{in}\|_{L^2}$$

Hypocoercivity

For $\nu > 0$, the equations look like

$$\partial_t \omega + L_\nu \omega = 0, \qquad L_\nu = B + \nu A^* A,$$

where $B = -B^*$, and $A = A^*$. To fix ideas (in L^2):

$$B = u(y)\partial_x, \qquad A = \partial_y$$

GOALS

- Prove enhanced dissipation $\|\omega(t)\|_{L^2} \lesssim \mathrm{e}^{-\lambda_\nu t}\|\omega^{in}\|_{L^2}$, with $\lambda_\nu \gg \nu$.
- Prove quantitative hypoellipticity

Key observation: the commutator $C_1 := [A, B] = u'\partial_x$ is nonzero, but $[C_1, B] = 0$.

Energy identities

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \nu\|\partial_y\omega\|_{L^2}^2 = 0,$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_y\omega\|_{L^2}^2 + \nu\|\partial_{yy}\omega\|_{L^2}^2 + \langle u'\partial_x\omega, \partial_y\omega\rangle_{L^2} = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u'\partial_x\omega, \partial_y\omega\rangle_{L^2} + \|u'\partial_x\omega\|_{L^2}^2 = -2\nu\langle\partial_{yy}\omega, u'\partial_{xy}\omega\rangle_{L^2} - \nu\langle\partial_{yy}\omega, u''\partial_x\omega\rangle_{L^2}.$$

For appropriately chosen α, β, the functional

$$\Phi_k = \frac{1}{2}\left[\|\omega_k\|_{L^2}^2 + \alpha\|\partial_y\omega_k\|_{L^2}^2 + 2\beta\left\langle\partial_x\omega_k, \partial_y\omega_k\right\rangle_{L^2}\right],$$

is coercive and satisfies (if $|u'| \geq \delta > 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_k + \varepsilon_0\nu^{1/3}|k|^{2/3}\Phi_k \leq 0 \qquad \Rightarrow \qquad \|\omega_k(t)\|_{L^2} \lesssim \|\omega_k^{in}\|_{L^2}\mathrm{e}^{-\varepsilon_0\nu^{\frac{1}{3}}|k|^{\frac{2}{3}}t}$$

General passive scalars

General passive scalar driven by shears: $u'\partial_x$ may be degenerate

- $\nu = 0$: $\|\nabla\psi\| \lesssim \langle t\rangle^{-\frac{1}{n}}$, where n is the order of vanishing of u' (Bedrossian, CZ '15). Key observation:

$$\omega_k(t) = \mathrm{e}^{-iku(y)t}\omega_k^{in}$$

- $\nu > 0$: $\|\omega_k(t)\|_{L^2} \lesssim \|\omega_k^{in}\|_{L^2}\mathrm{e}^{-\varepsilon_0\nu^{\frac{n}{n+2}}|k|^{\frac{2}{n+2}}t}$ (Bedrossian, CZ '15) Key observation: for any $\sigma \in [0,1]$, there holds

$$\sigma^{\frac{n-1}{n}}\|\omega_k\|_{L^2}^2 \lesssim \sigma\|\partial_y\omega_k\|_{L^2}^2 + \|u'(y)\omega_k\|_{L^2}^2$$

- monotone shears: $\nu \geq 0$: $\|\nabla\psi\|_{L^2} \lesssim \frac{\mathrm{e}^{-\varepsilon_0\nu^{1/3}t}}{\langle t\rangle}(\|\omega^{in}\|_{L^2} + \|\partial_y\omega^{in}\|_{L^2})$ (CZ '19) Key observation: combine vector field & hypocoercivity:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_k[\omega_k] + \delta_0\Phi_k[J_t\omega_k]) + \varepsilon_0\nu^{1/3}|k|^{2/3}(\Phi_k[\omega_k] + \delta_0\Phi_k[J_t\omega_k]) \leq 0$$

Euler/Navier-Stokes

- Enhanced dissipation for Navier-Stokes near shear flows
  - Poiseuille ($u(y) = y^2$): $\|\omega_k(t)\|_{L^2} \lesssim \|\omega_k^{in}\|_{L^2} \mathrm{e}^{-\varepsilon_0 \nu^{1/2}|k|^{1/2} t}$ (CZ, Elgindi, Widmayer '19), with nonlinear stability threshold $\nu^{3/4}$. Improved to $\nu^{2/3}$ (Del Zotto '21). Key observation:

$$\Phi_k = \frac{1}{2}[\|\omega_k\|_{L^2}^2 + \alpha \nu t \|\nabla \omega_k\|_{L^2}^2 + 4\beta \nu t^2 \langle y \partial_x \omega_k, \partial_y \omega_k \rangle_{L^2} + \gamma \nu t^3 \|y \partial_x \omega_k\|_{L^2}^2 + 2\gamma \nu t^3 \|\nabla \partial_x \psi\|_{L^2}^2].$$

  - Kolmogorov ($u(y) = \sin y$): same as above (Wei, Zhang, Zhao '19).
- Inviscid damping in 2d Euler?
  - Many linear results: monotone flows and symmetric flows with simple critical points (Zillinger, Wei/Zhang/Zhao, Jia ...)
  - Nonlinear results: monotone shears (Ionescu/Jia, Masmoudi/Zhao '21)
- Open question: inviscid damping by vector field in 2d Euler?

Models from kinetic theory

Equilibria in kinetic theory

- Vlasov-Poisson near an equilibrium $G = G(v)$ on $\mathbb{T}^d_x \times \mathbb{R}^d_v$:

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v G = \mathcal{N}, \qquad \nabla_x \cdot E = \rho,$$

  [Mouhot/Villani '09]

- Vlasov-Poisson-Fokker-Planck (weak collisions) near $\mu(v) = \mathrm{e}^{-|v|^2}$

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = \nu(\Delta_v f + \nabla_v \cdot (vf)) + \mathcal{N},$$

  [Bedrossian '17]

- Vlasov-Poisson-Landau on $\mathbb{T}^3_x \times \mathbb{R}^3_v$

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = \nu L f + \mathcal{N},$$

  [Chaturvedi/Luk/Nguyen '21]

- Boltzmann (weak collisions) on $\mathbb{T}^3_x \times \mathbb{R}^3_v$ or $\mathbb{R}^3_x \times \mathbb{R}^3_v$

$$\partial_t f + v \cdot \nabla_x f = \nu L + \nu \mathcal{N},$$

  [Bedrossian/CZ/Dolce '22]

Novelties

Taylor dispersion: Take

$$\partial_t f + v \cdot \nabla_x f = \nu(\Delta_v f + \nabla_v \cdot (vf))$$

on $\mathbb{R}^3_x \times \mathbb{R}^3_v$. Then [CZ/Gallay '21] we have $\|f_k(t)\|_{L^2_v} \lesssim \mathrm{e}^{-\lambda_{\nu,k} t}\|f_k(0)\|_{L^2_v}$

$$\lambda_{\nu,k} = \varepsilon_0 \begin{cases} \nu^{1/3}|k|^{2/3}, & \nu|k|^{-1} \leq 1 \\ |k|^2/\nu, & \nu|k|^{-1} \geq 1 \end{cases}$$

Key features in Boltzmann:

- No echoes
- The linearized operator L has kernel $\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$
- $\langle Lf, f\rangle$ controls $\| \langle v\rangle^{\gamma/2} f\|^2_{H^s}$, a weighted fractional derivative of order $s \in (0,1)$, away from the kernel. Can treat both hard $(\gamma + 2s > 0)$ and soft $(\gamma + 2s \leq 0)$ potentials.

Vector field & hypocoercivity

For linear passive scalar with fractional diffusion

$$\partial_t f + v \cdot \nabla_x f = -\nu(-\Delta_v)^s f$$

- The natural vector field is $J = \nabla_v + t\nabla_x$, which commutes with diffusion!
- The natural hypocoercivity functional is

$$\Phi = \frac{1}{2}\left[\|f\|^2 + a_{\nu,k}\|\nabla_v f\|^2 + b_{\nu,k}\langle \nabla_v f, \nabla_x f\rangle\right]$$

Then $\|f_k(t)\|_{L^2_v} \lesssim \mathrm{e}^{-\lambda_{\nu,k} t}\|f_k(0)\|_{L^2_v}$ where

$$\lambda_{\nu,k} = \varepsilon_0 \begin{cases} \nu^{\frac{1}{1+2s}}|k|^{\frac{2s}{1+2s}}, & \nu|k|^{-1} \leq 1 \\ |k|^2/\nu, & \nu|k|^{-1} \geq 1 \end{cases}$$

The case of Boltzmann

For Boltzmann (soft potential), enhanced dissipation regime:

$$\Phi_{M,k} = \frac{1}{2}\sum_{|\beta|\le N} C_\beta \Bigg(\left\| \langle v\rangle^{M} J^\beta f_k \right\|_{L^2_v}^2 + a_{\nu,k} \left\| \langle v\rangle^{M+\gamma/2} \nabla_v J^\beta f_k \right\|_{L^2_v}^2 + 2b_{\nu,k} \mathrm{Re} \left\langle \langle v\rangle^{M+\gamma/2} J^\beta (\nabla_x f)_k, \langle v\rangle^{M+\gamma/2} \nabla_v J^\beta f_k \right\rangle \Bigg)$$

For Boltzmann (soft potential), Taylor dispersion regime:

$$E^0_{M,k} = \frac{1}{2}\left(\|J^\beta f_k\|^2_{L^2} + \kappa_1 \|\langle v\rangle^{M} (I-P) J^\beta f_k\|^2_{L^2} + 2\kappa_2 E_{macro} \right)$$

Conclusion: enhanced dissipation, Taylor dispersion and "Landau damping" for macroscopic quantities. True at the nonlinear level for small perturbations of size in ε in Sobolev.

Other models

- Active suspensions near isotropic equilibria [CZ/Dietert/Gerard-Varet '22], $x \in \mathbb{T}^3$, $\boldsymbol{p} \in \mathbb{S}^2$

$$\begin{aligned} &\partial_t \psi + \boldsymbol{p} \cdot \nabla_x \psi - \frac{3\gamma}{4\pi} \boldsymbol{p} \otimes \boldsymbol{p} : E(\boldsymbol{u}) = \nu \Delta_{\boldsymbol{p}} \psi, \\ &- \Delta_x \boldsymbol{u} + \nabla_x q = \nabla_x \cdot \alpha \int_{\mathbb{S}^2} \psi(t, x, \boldsymbol{p}) \, \boldsymbol{p} \otimes \boldsymbol{p} \, \mathrm{d}\boldsymbol{p}, \\ &\nabla_x \cdot \boldsymbol{u} = 0. \end{aligned}$$

- Active Brownian particles near homogeneous equilibria, $\boldsymbol{p} \in \mathbb{S}^1$, $x \in \mathbb{T}^2$ [Bruna/Burger/Esposito/Schulz '21]

$$\partial_t f + (1 - \phi) \boldsymbol{p} \cdot \nabla_x f - \frac{\phi}{2\pi} \boldsymbol{p} \cdot \nabla_x \rho = \nu \partial_{\theta\theta} f.$$

THANK YOU