MEAN FIELD GAMES AND PDE'S IN INFINITE D.

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SUMMARY

1. INTRODUCTION

- DYNAMICAL EQUILIBRIUM MODELS FOR LARGE POPULATIONS (in some cases limits of N player Nash equilibria)
- INTRINSICALLY ∞D MODELS state of the population is a nonnegative bounded measure $\mathcal{M}$
- Finite D if finite state space or no common randomness
- EXAMPLES EXISTED IN ECON but simple except KRUSELL-SMITH
- EARLY 00's (2006 publication) J-M. Lasry-PL^2 MFG + monotonicity + MASTER EQ.
- LOTS OF MATH MATERIAL (3-4 meeting every year)
- MATHS, ENGINEERS, PHYSICISTS, ECO., FINANCE, COMPUTER SC.

2. ∞D EQUATIONS

- state of the population $m \in \mathcal{M}$
- state of an agent $x \in E$ (ex. $\mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d)$)
- "value function" $U(x, m, t) \in \mathbb{R}$ (ex. $t \in [0, T_0]$, $T_0 < \infty$)
- General form $(t \to T - t)$

$$\frac{\partial U}{\partial t} + D + T + N = 0$$

D : Decision block
T : Transport block
N : Noise block
a generic agent chooses a strategy which affects the value $U(D)$, those decision transport the measure $m(T)$, possibly in random settings (N)

$$U \mid_{t=0} = U_0(x, m)$$

Example: Finite horizon, single population, decision "à la Bellman", no intertemporal preference rate, continuous state space, Brownian noises...

$$D = \sup_{\alpha \in A} \{-b_\alpha(x,m).\partial_x U - f_\alpha(x,m)\}$$
$$\alpha^* \max, \quad b^* = b_{\alpha^*}(x,m)$$

$$T = \langle \partial_m U, -\partial_x.(b^* m) \rangle$$

$$N = -(\tfrac{\nu+\mu}{2}) \; \Delta_x U + \langle \partial_m U, -(\tfrac{\nu+\mu}{2}) \; \Delta_x m \rangle + \mu \langle \partial_x \partial_m U, .\partial_x U \rangle$$

ν variance of idiosynchratic noise
μ variance of common noise
$\partial_t U + A(x,m,\partial_x U) + \langle \partial_m U, -\partial_x.(B(x,m,\partial_x U)m) \rangle + N + 0$

RKS:
- KS model!
- see article Nobel Meeting: 150 years of Game Theory

3. ANALYSIS ON $\mathcal{M}(\times\ni)$

total size pop. fixed $\mathcal{P}$, 3 choices

- calculus in the vector space $\mathcal{M}$
- calculus in $(\mathcal{M}, W_1), \mathcal{P}(W_1$ or $W_2)$
- calculus on $\mathcal{P}_2$ or H limit of calculus on $(\mathbb{R}^d)^N(/\mathcal{S}^N)$ as $N \to +\infty$

Here $(x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ identified with all permutations
$\iff m = \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in \mathcal{P}$
$\iff X, R.V.s.t.X = x_i$ with proba $1/N$

Facts: PDE's with symmetry lift to $\mathcal{P}_2$ or H, "Lip f$^{\text{ions}}$ on "$(\mathbb{R}^d)^N/\mathcal{S}^N$)" become Lip f$^{\text{ions}}$ on H and Lip f$^{\text{ions}}$ on $\mathcal{P}, \phi(X) = \phi(m)(\mathcal{L}(X) = m)$ and $\|\partial_m \phi\|^2$ is

$$E|\nabla_x \phi|^2 = \int |\nabla_x \frac{\partial \phi}{\partial m}|^2 m$$

Master Eq. on $\mathcal{P}$ or on H : $\nabla_x U(X, \mathcal{L}(X)) = \tilde{U}(X)$

$$\frac{\partial \tilde{U}}{\partial t} + (A(X, \tilde{U}).\partial_X)\tilde{U} = B(X, \tilde{U}) - N$$

no regularization (few dim., "in m") Hilbertian formulation

Rk: Finite D

- Finite state space : $U\mathbb{R}^k \to \mathbb{R}^k$

$$\frac{\partial U}{\partial t} + (A(y, U).\partial_y)U = B(y, U) - N$$

  non conservative hyperbolic system with (very) partial regularization

- No common noise ($\mu = 0$) $U(x, m(t), t) = u(x, t)$

$$\frac{\partial u}{\partial t} + A(x, \partial_x u, m) - \frac{\nu}{2}\Delta_x u = 0, u\mid_{t=0} = u_0$$

$$\frac{\partial u}{\partial t} + \partial_x.(B(x\partial_x u, m) + \frac{\nu}{2}\Delta m = 0, m\mid_{t=T} = m_0$$

4. STATE OF THE ART

"SMOOTH" DATA

- PL^2 course: locally well-posed, globally under some monotonicity condition

Ex.: (B, A) mon on $\mathbb{R}^{2k}$, U_0 mon on $\mathbb{R}^k$ Lip. $\exists!$ U Lip., mon.

$$\frac{\partial U}{\partial t} + (A(y, U).\partial_y)U = B(y, U)$$

- in general non uniqueness but
- Ch. Bertucci-PL^2: if U_0 Lip ; $\partial_x U_0(x, m)$ Lip on $\mathbb{R}^d \times \mathcal{P}_1$

THEN $\exists T_0 \leqslant +\infty$ s.t. $\exists$! Lip. sol. on $[0, T](\forall 1 < T_0)$ thus Lip implies uniqueness

- IF $T \uparrow$, uniqueness is lost (possibly) when regularity is lost!

5. A DREAM?

- BEYOND $T_0 < \infty$?
- SET OF SOLUTIONS (definition?), AND RULES FOR THE GAME TO SELECT ONE SOL. ?
- numerical existence for the oil production model EDMOND (Y. Achdou, Ch. Bertucci, J-M. Lasry, PL^2, A. Rostand and J. Scheinkman)
- particular case: B. Seeger-PL^2 with decreasing data

$$U_0(y_1, \dots, y_N) \searrow \text{ in } y_i \ \forall i.$$