Stability of Schwarzschild for the spherically symmetric Einstein–massless Vlasov system

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Outline of the talk

General Relativity

General relativity is a geometric theory of *gravitation* whose main object of study are the *Lorentzian manifolds* $(\mathcal{M}^{1+3}, g, f)$ satisfying the *Einstein field equations*

$$Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

where $T_{\mu\nu}$ is the *energy momentum tensor of* *matter*. *Ex Naturally, we are interested in the *Einstein vacuum equations*

$$R_{\mu\nu} = 0. \tag{2}$$

Minkowski $\quad g_0 = -dt^2 + dr^2 + r^2 d\gamma_{S^2}$

Kerr $\quad g_{a,M}$

Schwarzschild $\quad g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\gamma_{S^2}$

The stability problem in general relativity

The dynamic nature of the EVE become apparent when the system is formulated as a *Cauchy problem*.

Theorem (Choquet-Bruhat)

The Einstein vacuum equations are well-posed in Sobolev regularity.*

Question: Is Minkowski/Schwarzschild/Kerr *stable** as a solution of the EVE?

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Schwarzschild	K-S, DHRT
Kerr	K-S

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Conjecture: The subextremal family of Kerr black holes is asymptotically stable as a solution of the EVE. * Massless Vlasov.

Collisionless many-particle systems in GR

We introduce a *distribution function* $f : \mathcal{P} \to [0, \infty)$ defined in the manifold

$$\mathcal{P} := \Big\{ (x,p) \in T\mathcal{M} : g_x(p,p) = 0, \text{ where } p \text{ is future-directed} \Big\}. \qquad (3)$$

The distribution function is only supported on *null vectors*. We call $\mathcal{P}$ the *mass-shell*.

$$T_{(x,p)}TM = \mathcal{H}_{(x,p)} \oplus \mathcal{V}_{(x,p)} = \mathrm{span}\{\partial_{x^\mu} - p^\nu \Gamma^\lambda_{\nu\mu} \partial_{p^\lambda}\} \oplus \mathrm{span}\{\partial_{p^\mu}\}$$

$$\bar{g} \equiv \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

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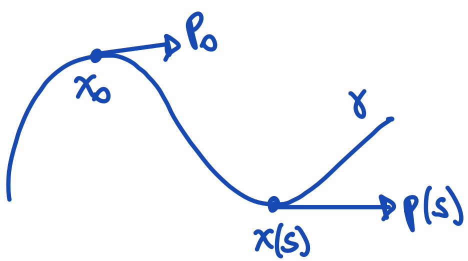
$$p^\alpha \partial_{x^\alpha} f - p^\alpha p^\beta \Gamma^i_{\alpha\beta} \partial_{p^i} f = 0. \tag{4}$$



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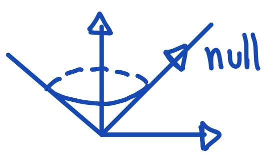
We define the *stress energy momentum tensor for massless Vlasov* by

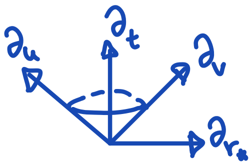


$$T_{\mu\nu}(x) := \int_{\mathcal{P}_x} f p_\mu p_\nu \, \mathrm{dvol}_{\mathcal{P}_x} \tag{5}$$

*dispersion.

The Einstein equations under spherical symmetry

Let $(\mathcal{M}^{3+1}, g)$ be a spherically symmetric spacetime in *double null coordinates* given by



$$g = -2\Omega^2(du \otimes dv + dv \otimes du) + r^2(u,v)d\gamma_{\mathbb{S}^2}, \tag{6}$$

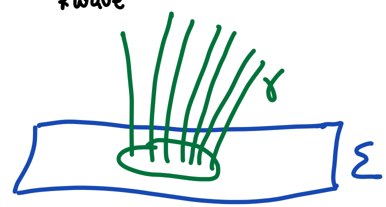
where Ω and r are non-negative functions. We introduce the *spherically symmetric Einstein–massless Vlasov system* by

* Galactic Dynamics, Plasma Physics

$$\begin{cases} \partial_u \partial_v r & = -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + 4\pi r T_{uv}, \\ \partial_u \partial_v \log \Omega & = \frac{\Omega^2}{4r^2} + \frac{\partial_u r \partial_v r}{r^2} - 8\pi T_{uv}, \\ \partial_u(\Omega^{-2}\partial_u r) & = -4\pi r T_{uu}\Omega^{-2}, \\ \partial_v(\Omega^{-2}\partial_v r) & = -4\pi r T_{vv}\Omega^{-2}, \end{cases} \tag{7}$$

* wave

* const



where T_{uu}, T_{uv} and T_{vv} are components of the energy momentum tensor.

Literature review

1. Stability of Minkowski for the spherically symmetric Einstein–massless Vlasov system (Dafermos).

2. Stability of Minkowski for the full Einstein–massless Vlasov system (Taylor, Bigorgne-Fajman-Joudioux-Smulevici-Thaller).
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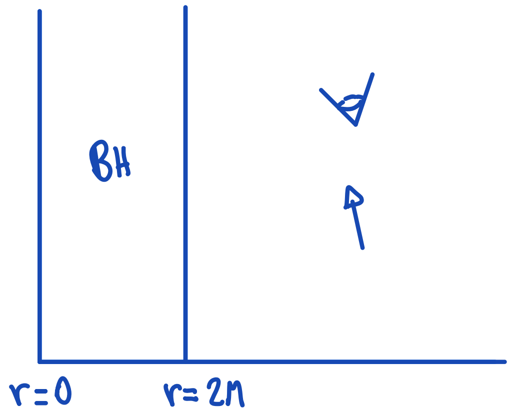
   *Jacobi, gauge *spt, gauge

3. Integrated energy decay for the massless Vlasov equation in slowly rotating Kerr (Andersson-Blue-Joudioux).
   *Deg, post AB for $\Box$.

4. Superpolynomial decay for the massless Vlasov equation in Schwarzschild (Bigorgne).
   *r^p, no ∂_p

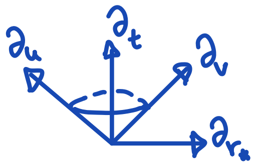
Asymptotic stability of Schwarzschild

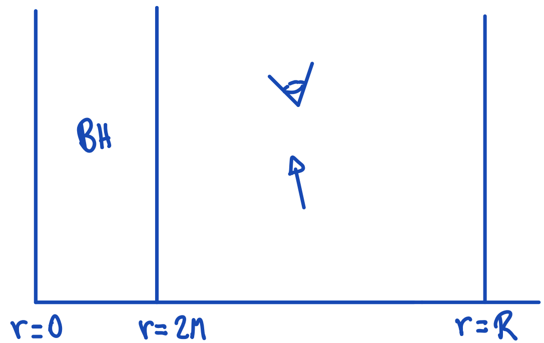


Theorem (VR)

The exterior of the Schwarzschild family is asymptotically stable as a solution of the spherically symmetric Einstein–massless Vlasov system. More precisely, for every initial data sufficiently close to Schwarzschild, the resulting solution asymptotes *exponentially* to another member of the Schwarzschild family.

The main result: linear version





Theorem (Decay of the stress energy momentum tensor)

Let f_0 be a compactly supported initial data for the massless Vlasov equation in Schwarzschild. There exists a positive constant $R > 2M$ such that the solution f of the massless Vlasov equation in Schwarzschild satisfies

$$T_{vv} \leq \frac{C_1}{r_*^6 \exp(C_2 u)}, \quad T_{uv} \leq \frac{C_1}{r^4 \exp(C_2 u)}, \quad T_{uu} \leq \frac{C_1}{r^2 \exp(C_2 u)}, \tag{8}$$

for all $(u, v) \in \{r \geq R\}$, where C_1 and C_2 are two positive constants depending on f_0, M and R. $*\partial T_1$ weight

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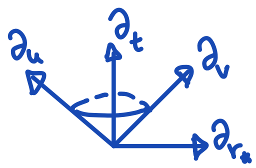

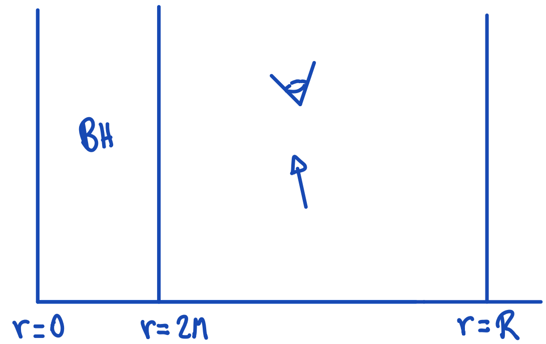

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$$T_{vv} \leq \frac{C_3}{\exp(C_4 v)}, \quad T_{uv} \leq \frac{C_3(1-\frac{2M}{r})}{\exp(C_4 v)}*, \quad T_{uu} \leq \frac{C_3(1-\frac{2M}{r})^2}{\exp(C_4 v)}, \tag{9}$$

for all $(u, v) \in \{r \leq R\}$, where C_3 and C_4 are two positive constants depending on f_0, M and R. *∂T

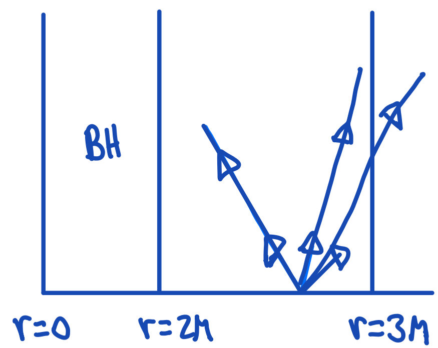
The geodesic flow in Schwarzschild I

The geodesic equations for the null momentum coordinates are given by

$$\begin{cases} \frac{dp^u}{ds} &= \frac{2M}{r^2}(p^u)^2 - \frac{l^2}{2r^3}, \\ \frac{dp^v}{ds} &= -\frac{2M}{r^2}(p^v)^2 + \frac{l^2}{2r^3}, \\ \frac{dl}{ds} &= 0, \end{cases} \tag{10}$$

where $l^2 := r^4\gamma_{AB}p^Ap^B_{*\mathrm{sph}}$ is a conserved quantity along the flow, the so-called *angular momentum*. We obtain another conserved quantity along the flow given by the *energy* $E := (1-\frac{2M}{r})(p^u+p^v)_{*T}$ since Schwarzschild is stationary.

$$T = \int_{P_x} f p \cdot p \cdot d\mathrm{vol}(p)$$



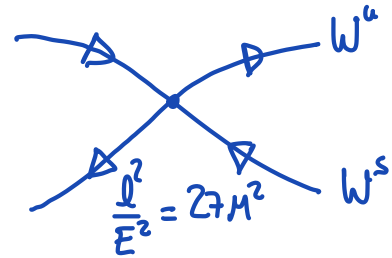
$$\frac{l^2}{E^2} - 27M^2$$

The geodesic flow in Schwarzschild II

The geodesic equation for the radial coordinate is given by

$$\begin{cases} \dot{r} & = p^r, \\ \dot{p}^r & = \dfrac{l^2}{r^4}(r - 3M), \end{cases}$$

*aut, decouple

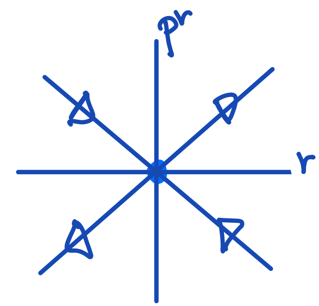


which admits a fixed point corresponding to the unique sphere where null geodesics can orbit, the so-called photon sphere. Linearizing around the fixed point, we obtain the system

*unstable trapp, exp decay

$$\begin{cases} \dot{r} & = p^r, \\ \dot{p}^r & = \dfrac{l^2}{81M^4}(r - 3M), \end{cases}$$

*Deg



which admits an hyperbolic fixed point. *stable manif.

Decay of the energy momentum tensor

Let us consider a fixed component of the stress energy momentum tensor of matter given by

$$T_{uv}(u,v) = \frac{\pi}{2r^2}\int_{\mathbb{R}+}\int_{\mathbb{R}+}(\Omega^2 p^u)(\Omega^2 p^v) f \frac{dp^v}{p^v} l dl_{\text{vol}}. \tag{13}$$

The decay estimates for T_{uv} come from several features of the geodesic flow in Schwarzschild:

1. The red-shift — $\frac{dp^v}{ds} + \frac{2M}{r^2}(p^v)^2 = \frac{l^2}{2r^3}$
2. Future trapped geodesics — p^r
3. Decay towards null infinity — $4\left(1-\frac{2M}{r}\right)p^u p^v = \frac{l^2}{r^2}$

Derivatives of the energy momentum tensor I

To estimate radial derivatives of the energy momentum tensor like

$$\partial_r T_{uv} = \frac{\pi}{2r^2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (\Omega^2 p^u)(\Omega^2 p^v) \partial_r f \frac{dp^v}{p^v} l dl + Err,$$

we require bounds for $\partial_r f$. For this purpose we estimate <u>Jacobi fields in the mass-shell</u>.

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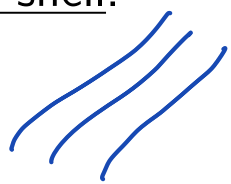


$J(t) = \frac{\partial \gamma_\tau}{\partial \tau}\Big|_{\tau=0}$, $\quad \nabla_{\dot\gamma}\nabla_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma$

Let $V \in T\mathcal{P}$ be an arbitrary vector field on the mass-shell. By the Vlasov equation, we have

$$V(f)(x_s, p_s) = J(f)(x_0, p_0), \tag{14}$$

*grow
shrink

where $J := d\phi_{-s}|_{(x_s,p_s)}(V)$ is a *Jacobi field in the mass shell* along the corresponding null geodesic γ.

Jacobi fields along the photon sphere

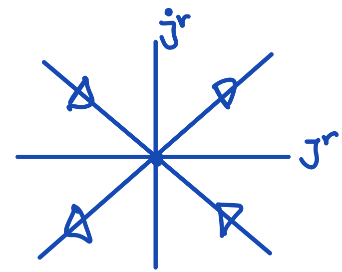
Let γ be a trapped null geodesic contained in the photon sphere. We set a parallel vector field P_G along γ by

$$P_G := \frac{\partial_r}{\sqrt{3}} - \frac{s}{3\sqrt{3}M}\dot{\gamma}. \tag{15}$$

An explicit computation shows that the radial component of a Jacobi field $J = J^0\dot{\gamma} + J^G P_G$ satisfies the linear ode

$$\frac{d^2 J^G}{ds^2} = \frac{l^2}{81M^4} J^G.$$

* Match lineariz flow
* $K<0$



A similar computation in the mass-shell in terms of the Sasaki metric shows the <u>same</u> ode for the radial components of a Jacobi field

* $\partial_r f$ grows

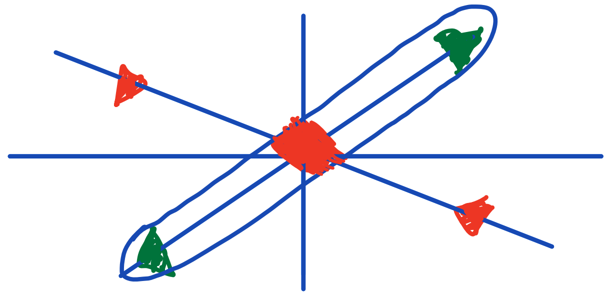
$$\hat{J} = J^0 \mathrm{Hor}_{(x,p)}(p) + J^G \mathrm{Hor}_{(x,p)}(P_G) + \dot{J}^G_{(x,p)} \mathrm{Ver}(P_G).$$

Derivatives of the energy momentum tensor II

Let us investigate the value on the photon sphere of the term in the radial derivative $\partial_r T_{uv}$

$$\frac{\pi}{2r^2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}(\Omega^2 p^u)(\Omega^2 p^v)Vf\frac{dp^v}{p^v}ldl\bigg|_{r=3m}. \tag{16}$$

By the previous computation, Jacobi fields along trapped null geodesics grow or <u>shrink</u> exponentially fast.



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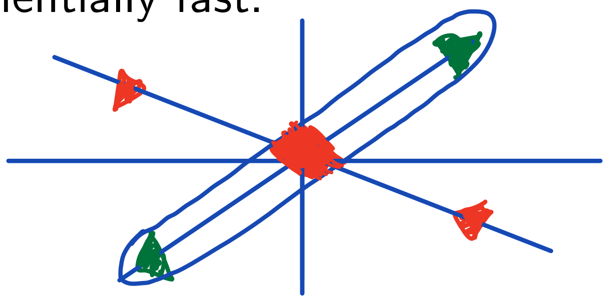

Nonetheless, the set of Jacobi fields shrinking exponentially fast are <u>concentrated</u> in a small region of $\mathcal{P}_{x_{*\partial\tau}}$!

The nonlinear difficulties I

The result follows via a boostrap argument including exponential decay for $T_{\mu\nu}$ and $\partial_r T_{\mu\nu}$ in the bootstrap assumptions.

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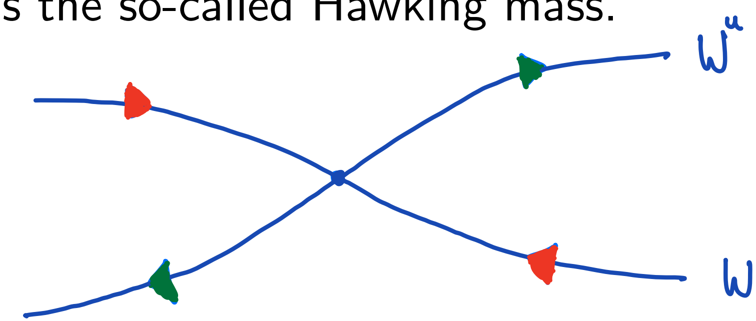
Let us focus in the null geodesic flow around $r = 3m$. The geodesic equation for the area-radius is given by

$$\begin{cases} \dot{r} &= p^r, \\ \dot{p}^r &= \dfrac{l_*^2}{r^4}(r - 3m) - 4\pi r\Big(T_{uu}(p^u)^2 - 2T_{uv}p^u p^v + T_{vv}(p^v)^2\Big)_*, \end{cases} \tag{17}$$

where $m(u, v)$ is the so-called Hawking mass.



$$\frac{l^2}{E^2(s)} = 27m_*^2$$

*teleology

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where $m(u, v)$ is the so-called Hawking mass. Although T is not Killing anymore, we can still work with the *energy of a geodesic* γ

$$E(s) := -g(T, \dot{\gamma}) = -\partial_u r p^u(s) + \partial_v r p^v(s)_{*\dot{E}} \tag{18}$$

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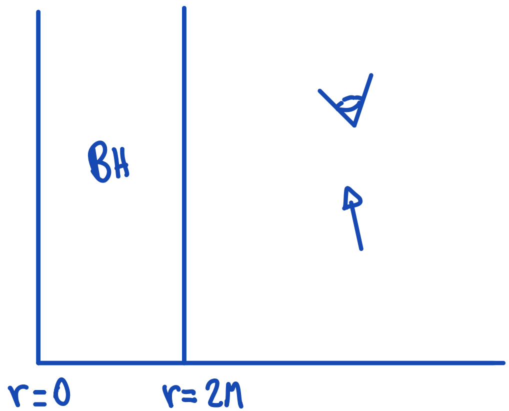
is asymptotically parallel. Moreover, the component J^G of some Jacobi fields J written using a suitable double null frame satisfies

$$\frac{d^2 J^G}{ds^2} = \frac{l^2}{81m^4} J^G + Err.$$

A similar ode is satisfied by the G-components of the corresponding Jacobi fields in the mass-shell. Estimates for the components of the Jacobi fields in spacetime can be recovered integrating the Jacobi equation.

Asymptotic stability of Schwarzschild



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