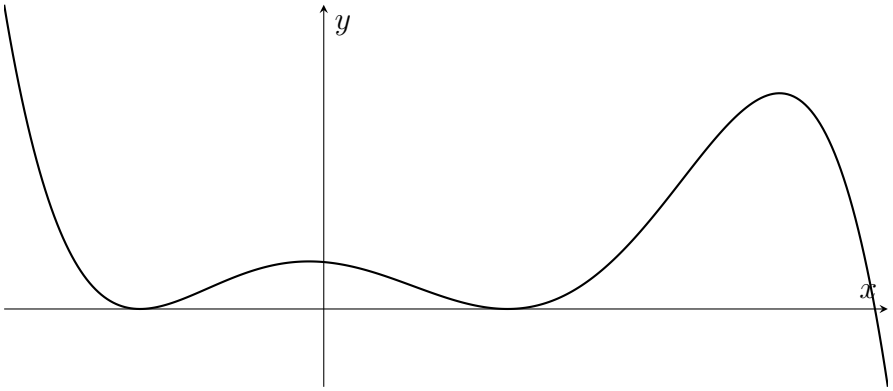
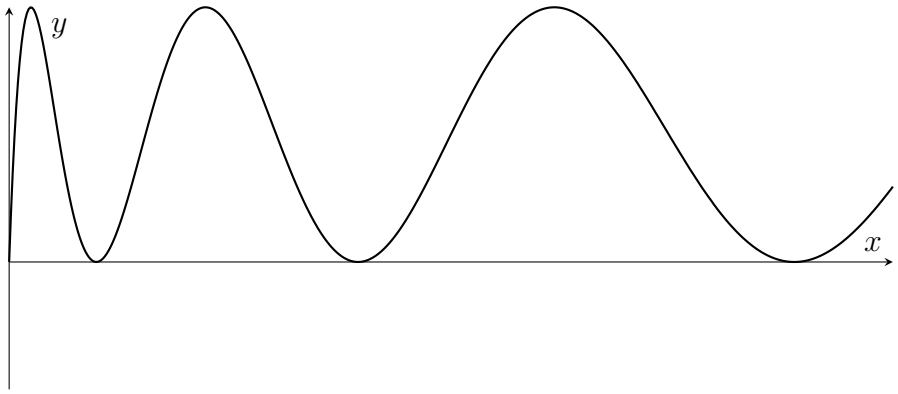
Warm-up (based on MAT 2012 Q1E, 2010 Q1D, 2007 Q1E)
Which one of the following equations could possibly have this graph?



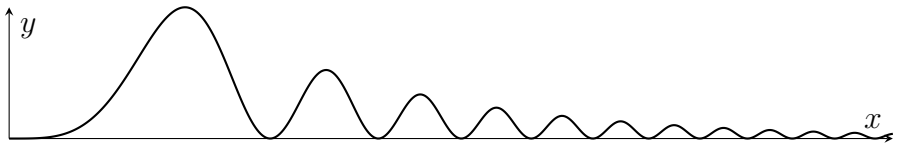
(a) $y = (3-x)^2(3+x)^2(1-x)$, (b) $y = -x^2\,(x-9)(x^2-3)$,
(c) $y = (x-6)(x-2)^2(x+2)^2$, (d) $y = (x^2-1)^2(3-x)$,
(e) $y = (x+1)(x-1)(x-3)$.

Which one of the following equations could possibly have this graph?



(a) $y = \sin(\sqrt{x})$, (b) $y = \sin^2(\sqrt{x})$, (c) $y = \sin(x^2)$,
(d) $y = \sin^2(x^2)$, (e) $y = \sin(e^x)$.

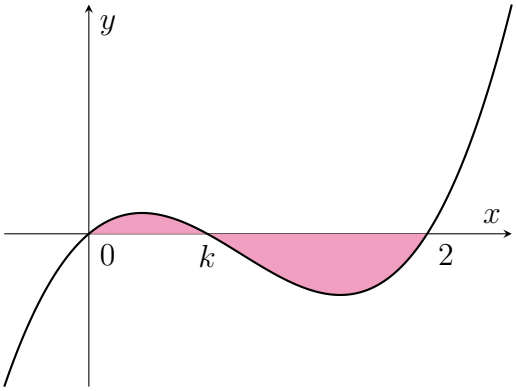
Which one of the following equations could possibly have this graph?



(a) $y = 2^{-x}\sin(x)$, (b) $y = 2^{-x}\sin^2(x)$, (c) $y = 2^{-x}\sin(x^2)$,
(d) $y = 2^{-x}\sin^2(x^2)$, (e) $y = 2^{-\sin(x)}$.

Extension Sketch graphs for the other 12 equations on this page.

Long question 1 (MAT 2013 Q3)
Let $0 < k < 2$. Below is sketched a graph of $y = f(x)$ where $f_k(x) = x(x-k)(x-2)$. Let $A(k)$ denote the area of the shaded region.



(i) Without evaluating them, write down an expression for $A(k)$ in terms of two integrals.

(ii) Explain why $A(k)$ is a polynomial in k of degree 4 or less. You are not required to calculate $A(k)$ explicitly.

(iii) Verify that $f_k(1+t) = -f_{2-k}(1-t)$ for any t.

(iv) How can the graph of $y = f_k(x)$ be transformed to the graph of $y = f_{2-k}(x)$?

Deduce that $A(k) = A(2-k)$.

(v) Explain why there are constants a, b, c such that

$$A(k) = a(k-1)^4 + b(k-1)^2 + c.$$

You are not required to calculate a, b, c explicitly.

Extension Calculate $A(k)$, and a, b, and c explicitly.

Long question 2 (Very slightly adapted from MAT 2012 Q5)

A particular robot has three commands;
F: Move forward a unit distance;
L: Turn left 90°
R: Turn right 90°

A *program* is a sequence of commands. We consider particular programs P_n (for $n \geqslant 0$) in this question. The basic program P_0 just instructs the robot to move forward:

$$P_0 = \mathbf{F}.$$

The program P_{n+1} (for $n \geqslant 0$) involves performing P_n, turning left, performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n \mathbf{L} P_n \mathbf{L} P_n \mathbf{R}.$$

So, for example, $P_1 = \mathbf{FLFR}$.

(i) Write down the program P_2.

(ii) How many **F** commands does the robot perform during the program P_n?

(iii) Let l_n be the total number of commands in P_n; so, for example, $l_0 = 1$ and $l_1 = 4$. Write down an equation relating l_{n+1} to l_n. Hence write down a formula for l_n in terms of n.
Hint: consider $l_n + 2$.

(iv) The robot starts at the origin, facing along the positive x-axis. What direction is the robot facing after performing the program P_n? Justify your answer.

(v) Draw the path the robot takes when it performs the program P_4.

(vi) Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$. Give an equation relating (x_{n+1}, y_{n+1}) to (x_n, y_n).

Where is the robot after performing P_8? Where is it after performing P_{8k}?

Bonus problem (not MAT)
When you roll two normal fair six-sided dice, the sum is one of the following totals with the following probabilities;

Total	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

I've got two six-sided dice, and each has a whole number greater than or equal to 1 on each side. My two dice are not the same, but they are each 'fair' in the sense that each of the six sides is equally likely to be rolled. When I roll my two dice, the sum might be any one of the same totals as above with the same probabilities as above. What are the numbers on my dice?

If you want to check your solution, search for "Sicherman dice"

Extension 3D-print these dice and use them to spice up Monopoly or D&D.