Warm-up (not MAT) The function $f(n)$ is defined for all positive integers n as follows; $f(1) = 1$ and, for all $n \geqslant 1$,

$$f(2n) = f(n), \quad f(2n+1) = f(n) + 1.$$

Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$.
What is $f(2^{100})$?
What does $f(n)$ have to do with the binary representation of n?

Now consider a new function $g(n)$ which is defined for all positive integers n as follows; $g(1) = 2$, and for all $n \geqslant 1$,

$$g(2n) = g(n) + 1, \quad g(2n+1) = g(n) + 2.$$

Find $g(n)$ for some small values of n and work out the link between $g(n)$ and the binary representation of n.

Short question 1 (MAT 2010 Q1G)
The function f, defined for whole positive numbers, satisfies $f(1) = 1$ and also the rules

$$f(2n) = 2f(n),$$
$$f(2n+1) = 4f(n),$$

for all values of n. How many numbers satisfy $f(n) = 16$?

(a) 3, (b) 4, (c) 5, (d) 6.

Extension: (MAT 2011 Q1J)
The function $f(n)$ is defined for positive integers n according to the rules

$$f(1) = 1, \qquad f(2n) = f(n), \qquad f(2n+1) = (f(n))^2 - 2.$$

The value of $f(1) + f(2) + f(3) + \cdots + f(100)$ is

(a) -86, (b) -31, (c) 12, (d) 58.

Extension: (MAT 2014 Q1H)
The function $F(n)$ is defined for all positive integers as follows: $F(1) = 0$ and for all $n \geqslant 2$,

$$\begin{aligned} F(n) &= F(n-1) + 2 && \text{if 2 divides } n \text{ but 3 does not divide } n; \\ F(n) &= F(n-1) + 2 && \text{if 3 divides } n \text{ but 2 does not divide } n; \\ F(n) &= F(n-1) + 2 && \text{if 2 and 3 both divide } n; \\ F(n) &= F(n-1) && \text{if neither 2 nor 3 divides } n. \end{aligned}$$

The value of $F(6000)$ equals

(a) 9827, (b) 10121, (c) 11000, (d) 12300, (e) 12352.

Short question 2 (MAT 2012 Q1B)
Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers. Then N will definitely be a square number whenever

(a) k is even,

(b) $k + n$ is odd,

(c) k is odd but $m + n$ is even,

(d) $k + n$ is even.

Long question (MAT 2013 Q5)
We define the *digit sum* of a non-negative integer to be the sum of its digits. For example, the digit sum of 123 is $1 + 2 + 3 = 6$.

(i) How many positive integers less than 100 have digit sum equal to 8?

Let n be a positive integer with $n < 10$.

(ii) How many positive integers less than 100 have digit sum equal to n?

(iii) How many positive integers less than 1000 have digit sum equal to n?

(iv) How many positive integers between 500 and 999 have digit sum equal to 8?

(v) How many positive integers less than 1000 have digit sum equal to 8, and one digit at least 5?

(vi) What is the total of the digit sums of the integers from 0 to 999 inclusive?

Bonus question (not MAT)
Exactly one of the following numbers is prime. Which one? No calculators allowed!

$$91, \quad 3599, \quad 80111, \quad 125729, \quad 1010101, \quad 104060401 \quad 987654321.$$

Find a non-trivial factor of each of the other numbers (not 1 or the number itself).