Warm-up (not MAT)
Recall that $a^2 - b^2 = (a-b)(a+b)$.
Factorise $a^3 - b^3$.
Factorise $a^4 - b^4$.
If $p(x) = x^n - 1$, explain why $(x-1)$ divides $p(x)$ without remainder.
If $p(x) = x^n + 1$, when does $(x+1)$ divide $p(x)$ without remainder?

2015 Q2
(i) Expand and simplify

$$(a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n).$$

(ii) The prime number 3 has the property that it is one less than a square number. Are there any other prime numbers with this property? Justify your answer.

(iii) Find all the prime numbers that are one more than a cube number. Justify your answer.

(iv) Is $3^{2015} - 2^{2015}$ a prime number? Explain your reasoning carefully.

(v) Is there a positive integer k for which $k^3 + 2k^2 + 2k + 1$ is a cube number? Explain your reasoning carefully.

2011 Q2

Suppose that x satisfies the equation

$$x^3 = 2x + 1.$$

(i) Show that

$$x^4 = x + 2x^2 \qquad \text{and} \qquad x^5 = 2 + 4x + x^2.$$

(ii) For every integer $k \geqslant 0$, we can uniquely write

$$x^k = A_k + B_k x + C_k x^2$$

where A_k, B_k, C_k are integers. So, in part (i), it was shown that

$$A_4 = 0,\ B_4 = 1, C_4 = 2 \qquad \text{and} \qquad A_5 = 2,\ B_5 = 4,\ C_5 = 1.$$

Show that

$$A_{k+1} = C_k, \qquad B_{k+1} = A_k + 2C_k, \qquad C_{k+1} = B_k.$$

(iii) Let

$$D_k = A_k + C_k - B_k.$$

Show that $D_{k+1} = -D_k$ and hence that

$$A_k + C_k = B_k + (-1)^k.$$

(iv) Let $F_k = A_{k+1} + C_{k+1}$. Show that

$$F_k + F_{k+1} = F_{k+2}.$$

Extension: Try MAT 2017 Q2

2015 Q5

The following functions are defined for all integers a, b and c:

$$\begin{aligned} p(x) &= x+1 \\ m(x) &= x-1 \\ s(x,y,z) &= \begin{cases} y & \text{if } x \leqslant 0 \\ z & \text{if } x > 0. \end{cases} \end{aligned}$$

(i) Show that the value of

$$s\Big(\, s(p(0), m(0), m(m(0))),\;\; s(p(0), m(0), p(p(0))),\;\; s(m(0), p(0), m(p(0)))\,\Big)$$

is 2.

Let f be a function defined, for all integers a and b, as follows:

$$f(a,b) = s(\,b,\; p(a),\; p(f(a, m(b)))\,).$$

(ii) What is the value of $f(5,2)$?

(iii) Give a simple formula for the value of $f(a,b)$ for all integers a and all *positive* integers b, and explain why this formula holds.

(iv) Define a function $g(a,b)$ in a similar way to f, using only the functions p, m and s, so that the value of $g(a,b)$ is equal to the sum of a and b for all integers a and all integers $b \leqslant 0$.

Explain briefly why your function gives the correct value for all such values of a and b.

Bonus question (not MAT)
This is a game involving three players (P_1, P_2, and P_3) and three cakes. Initially each player has one cake.

Taking it in turns, starting with the first player P_1, each player chooses someone and redistributes their cake among the players. For example, I might choose someone else to give all of their cake to me. Or I might be nice and choose someone to split their cake in the ratio 70:20:10, giving the big piece to me, giving the middle-sized piece to the other player as a sort of bribe, and letting my victim keep 10% of their cake for themselves (or any other ratio). Here's another example of how the game might progress;

- Each player starts with one cake.
- P_1 decides (greedily) that P_3 should give their entire cake to P_1.
- P_1 now has two cakes, P_2 has one cake, and P_3 has no cake.
- P_2 sees that P_1 has the most cake, but is worried that P_3 will steal all their cake. They choose P_1 and take half a cake for themselves, but let P_1 keep the remaining $1\frac{1}{2}$ cakes.
- P_1 now has $1\frac{1}{2}$ cakes, P_2 has $\frac{1}{2}$ a cake, and P_3 has no cake.
- Finally, P_3 takes revenge on P_1 by choosing them and redistributing all their cake to P_3.
- At the end of the game, P_1 has no cake, P_2 has $\frac{1}{2}$ a cake, and P_3 has $1\frac{1}{2}$ cakes.

Each player is aware of the rules and would like to end the game with as much cake as possible. What should they do?

Extension: (very hard) What if there are more than 3 players?