2015 Q5

The following functions are defined for all integers a, b and c:

$$\begin{aligned} p(x) &= x+1 \\ m(x) &= x-1 \\ s(x,y,z) &= \begin{cases} y & \text{if } x \leqslant 0 \\ z & \text{if } x > 0. \end{cases} \end{aligned}$$

(i) Show that the value of

$$s\Big(\, s(p(0), m(0), m(m(0))),\ \ s(p(0), m(0), p(p(0))),\ \ s(m(0), p(0), m(p(0)))\,\Big)$$

is 2.

Let f be a function defined, for all integers a and b, as follows:

$$f(a,b) = s(\,b,\ p(a),\ p(f(a, m(b)))\,).$$

(ii) What is the value of $f(5,2)$?

(iii) Give a simple formula for the value of $f(a,b)$ for all integers a and all *positive* integers b, and explain why this formula holds.

(iv) Define a function $g(a,b)$ in a similar way to f, using only the functions p, m and s, so that the value of $g(a,b)$ is equal to the sum of a and b for all integers a and all integers $b \leqslant 0$.

Explain briefly why your function gives the correct value for all such values of a and b.

2010 Q5

This question concerns calendar dates of the form

$$d_1d_2/m_1m_2/y_1y_2y_3y_4$$

in the order day/month/year.

The question specifically concerns those dates which contain no repetitions of a digit. For example, the date 23/05/1967 is one such date but 07/12/1974 is not such a date as both $1 = m_1 = y_1$ and $7 = d_2 = y_3$ are repeated digits.

We will use the Gregorian Calendar throughout (this is the calendar system that is standard throughout most of the world; see below.)

(i) Show that there is no date with no repetition of digits in the years from 2000 to 2099.

(ii) What was the last date before today with no repetition of digits? Explain your answer.

(iii) When will the next such date be? Explain your answer.

(iv) How many such dates were there in years from 1900 to 1999? Explain your answer.

[The Gregorian Calendar uses 12 months, which have, respectively, 31, 28 or 29, 31, 30, 31, 30, 31, 31, 30, 31, 30 and 31 days. The second month (February) has 28 days in years that are not divisible by 4, or that are divisible by 100 but not 400 (such as 1900); it has 29 days in the other years (leap years).]

2013 Q1J

For a real number x we denote by $[x]$ the largest integer less than or equal to x.
Let n be a natural number. The integral

$$\int_0^n [2^x] \, \mathrm{d}x$$

equals

(a) $\log_2((2^n-1)!)$, (b) $n2^n - \log_2((2^n)!)$, (c) $n2^n$, (d) $\log_2((2^n)!)$,

where $k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer k.

Bonus question (not MAT)
You might have seen that the derivative of e^x is e^x. In other words, e^x is an example of a function $f(x)$ with $f'(x) = f(x)$.

The derivative of e^{kx} is ke^{kx}. Can you find two functions $f(x)$ and $g(x)$ with

$$f'(x) = g(x), \quad g'(x) = f(x) \quad ?$$

If you've seen the derivative of trigonometric functions, then you already know examples of four functions $f(x)$, $g(x)$, $h(x)$, $j(x)$ with

$$f'(x) = g(x), \quad g'(x) = h(x), \quad h'(x) = j(x), \quad j'(x) = f(x).$$

Can you find three functions $f(x)$, $g(x)$, $h(x)$ with

$$f'(x) = g(x), \quad g'(x) = h(x), \quad h'(x) = f(x) \quad ?$$