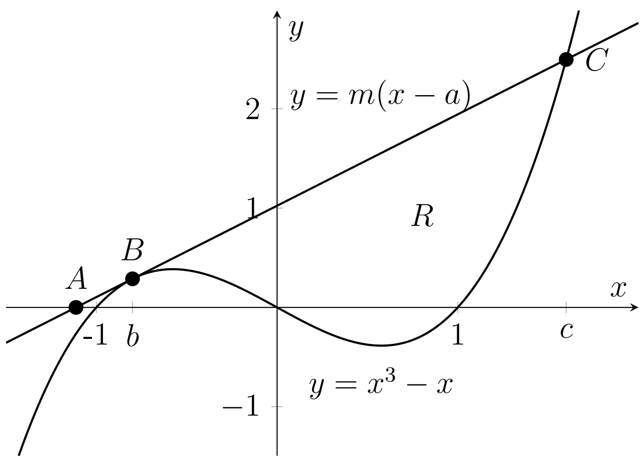
Last time we did parts (i) and (ii) of this question, but then we got stuck because we didn't know if $a < b$ or $a > b$. Today, we're going to do parts (iii), (iv), and (v), and assume that $b > a$, as in the diagram.

MAT 2011 Q3

The graphs of $y = x^3 - x$ and $y = m(x - a)$ are drawn on the axes below. Here $m > 0$ and $a \leqslant -1$.

The line $y = m(x - a)$ meets the x-axis at $A = (a, 0)$, *touches* the cubic $y = x^3 - x$ at B and intersects again with the cubic at C. The x-coordinates of B and C are respectively b and c.



(i) Use the fact that the line and cubic *touch* when $x = b$, to show that $m = 3b^2 - 1$.

(ii) Show further that

$$a = \frac{2b^3}{3b^2 - 1}.$$

(iii) If $a = -10^6$, what is the approximate value of b?

(iv) Using the fact that

$$x^3 - x - m(x - a) = (x - b)^2 (x - c)$$

(which you need not prove), show that $c = -2b$.

(v) R is the finite region bounded above by the line $y = m(x - a)$ and bounded below by the cubic $y = x^3 - x$. For what value of a is the area of R largest?

Show that the largest possible area of R is $\frac{27}{4}$.

MAT 2011 Q1D

The fraction of the interval $0 \leqslant x \leqslant 360°$, for which one (or both) of the inequalities

$$\sin x \geqslant \frac{1}{2}, \qquad \sin 2x \geqslant \frac{1}{2}$$

is true, equals

$$\text{(a)} \quad \frac{1}{3}, \qquad \text{(b)} \quad \frac{13}{24}, \qquad \text{(c)} \quad \frac{7}{12}, \qquad \text{(d)} \quad \frac{5}{8}.$$

MAT 2014 Q1E

As x varies over the real numbers, the largest value taken by the function

$$\left(4\sin^2 x + 4\cos x + 1\right)^2$$

equals

(a) $17 + 12\sqrt{2}$, (b) 36, (c) $48\sqrt{2}$, (d) $64 - 12\sqrt{3}$, (e) 81.

MAT 2007 Q1B

The greatest value which the function

$$f(x) = \left(3\sin^2(10x + 11) - 7\right)^2$$

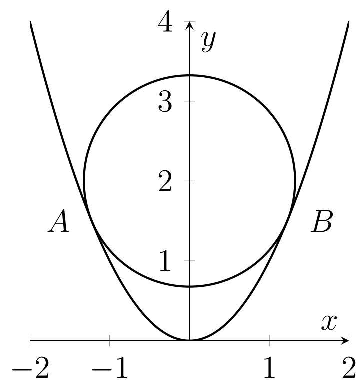
takes, as x varies over all real values, equals

(a) -9, (b) 16, (c) 49, (d) 100.

Note: this question uses $\cos^{-1} x$, the function which returns the angle θ in radians between 0 and π with $\cos\theta = x$. Since 2007, the A level syllabus has changed, and you might not have seen radians (so they're no longer on the MAT syllabus). If you want $\cos^{-1} x$ to return an angle in degrees, then your answer for part (iii) will be something like $\frac{7}{4}\frac{\pi}{180^\circ}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right)$

MAT 2012 Q4

The diagram below shows the parabola $y = x^2$ and a circle with centre $(0, 2)$ just 'resting' on the parabola. By 'resting' we mean that the circle and parabola are tangential to each other at the points A and B.



(i) Let (x, y) be a point on the parabola such that $x \neq 0$. Show that the gradient of the line joining this point to the centre of the circle is given by

$$\frac{x^2 - 2}{x}.$$

(ii) With the help of the result from part (i), or otherwise, show that the coordinates of B are given by

$$\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right).$$

(iii) Show that the area of the sector of the circle enclosed by the radius to A, the minor arc AB and the radius to B is equal to

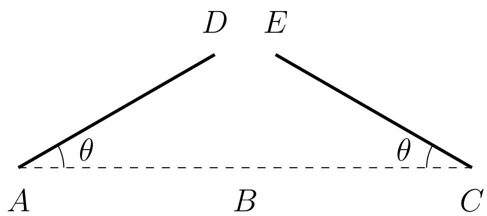
$$\frac{7}{4}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right).$$

(iv) Suppose now that a circle with centre $(0, a)$ is resting on the parabola, where $a > 0$. Find the range of values of a for which the circle and parabola touch at two distinct points.

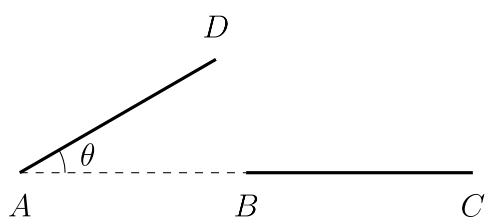
(v) Let r be the radius of a circle with centre $(0, a)$ that is resting on the parabola. Express a as a function of r, distinguishing between the cases in which the circle is, and is not, in contact with the vertex of the parabola.

Bonus question (not MAT)
It's a hot day and I'd like to open the windows a bit to get some airflow into the room. I've noticed that, if I open both windows a little, then the gap between the windows isn't actually that big; perhaps, counter-intuitively, I should only open one window? This geometry problem will explore that.

Here's a diagram of my windows (looking down from above).



(i) When they're closed, the windows are at AB and CB. The windows only open to an angle of θ, where $0 < \theta < 90°$, and when they're open they are at AD and CE. Find the length of DE in terms of θ; this is the gap between the windows when I open them both.



(ii) When the left window is open and the right window is closed, the windows are at AD and CB. Find the perpendicular distance from B to AD in terms of θ; this is the gap between the windows when I open one of them.

(iii) Find the range of θ for which the gap in part (i) is smaller than the gap in part (ii).