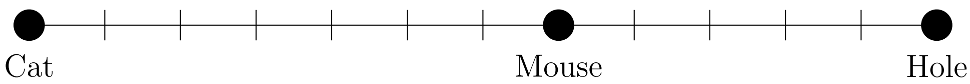
**MAT 2010 Q7**

In a game of *Cat and Mouse*, a cat starts at position 0, a mouse starts at position $m$ and the mouse's hole is at position $h$. Here $m$ and $h$ are integers with $0 < m < h$. By way of example, a starting position is shown below where $m = 7$ and $h = 12$.




With each turn of the game, one of the mouse or cat (but not both) advances one position towards the hole *on the condition that the cat is always strictly behind the mouse and never catches it.* The game ends when the mouse reaches the safety of its hole at position $h$.

This question is about calculating the number, $g(h, m)$, of different sequences of moves that make a game of Cat and Mouse.

Let $C$ denote a move of the cat and $M$ denote a move of the mouse. Then, for example, $g(3, 1) = 2$ as $MM$ and $MCM$ are the only possible games. Also $CMCCM$ is *not* a valid game when $h = 4$ and $m = 2$ as the mouse would be caught on the fourth turn.

(i) Write down the five valid games when $h = 4$ and $m = 2$.

(ii) Explain why $g(h, h-1) = h - 1$ for $h \geqslant 2$.

(iii) Explain why $g(h, 2) = g(h, 1)$ for $h \geqslant 3$.

(iv) By considering the possible first moves of a game, explain why

$$g(h, m) = g(h, m+1) + g(h-1, m-1) \qquad \text{when } 1 < m < h - 1.$$

(v) Below is a table with certain values of $g(h, m)$ filled in. Complete the remainder of the table and verify that $g(6, 1) = 42$.

| $m$ \ $h$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 5 | | | | | 5 |
| 4 | | | | 4 | |
| 3 | | | 3 | | |
| 2 | | 2 | | | |
| 1 | 1 | | | | |

**MAT 2007 Q5**

Let $f(n)$ be a function defined, for any integer $n \geqslant 0$, as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ (f(n/2))^2 & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 2f(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(i) What is the value of $f(5)$?

The *recursion depth* of $f(n)$ is defined to be the number of other integers $m$ such that the value of $f(m)$ is calculated whilst computing the value of $f(n)$. For example, the recursion depth of $f(4)$ is 3, because the values of $f(2)$, $f(1)$, and $f(0)$ need to be calculated on the way to computing the value of $f(4)$.

(ii) What is the recursion depth of $f(5)$?

Now let $g(n)$ be a function, defined for all integers $n \geqslant 0$, as follows:

$$g(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 + g(n/2) & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 1 + g(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(iii) What is $g(5)$?

(iv) What is $g\left(2^k\right)$, where $k \geqslant 0$ is an integer? Briefly explain your answer.

(v) What is $g\left(2^l + 2^k\right)$ where $l > k \geqslant 0$ are integers? Briefly explain your answer.

(vi) Explain briefly why the value of $g(n)$ is equal to the recursion depth of $f(n)$.

**Bonus question** (not MAT)
It's a remarkable fact that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$$

You might have already seen this; if so, take a moment to think about how strange this fact is.

I'd like you to use that fact to calculate

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x$$

[Hints: expand the numerator, use polynomial division to simplify the fraction into a quotient plus a remainder over $1+x^2$, then integrate term by term.]

This integral has its own Wikipedia page. Can you see why?

*I can't tell you the name of the Wikipedia page without giving the game away, but if you search Wikipedia for the phrase "requiring only elementary techniques" then it's the first result, if you want to read more about this.*