This week we're revisiting the bonus problems. These questions are not MAT questions, they're just extra bits of maths that I wanted to share with you.

Bonus problem (13 August)
I've been thinking about applying effects to a sine wave (which might cause your voice to sound like a robot). My idea is to do something to "lower the resolution" of the sine wave.

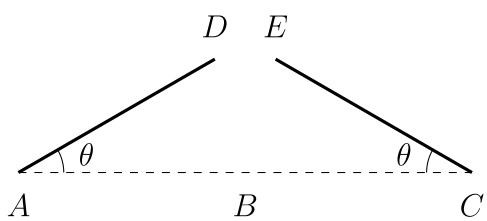
Let $[x]$ be the integer closest to x, so $[\pi] = 3$ and $[-2] = -2$. We'll round numbers midway between integers upwards, so $[-1.5] = -1$ and $[0.5] = 1$.

- Sketch $y = [\sin x]$
- Sketch $y = \dfrac{1}{3}[3\sin x]$
- Sketch $y = \dfrac{4}{3}\left[\dfrac{3}{4}\sin x\right]$

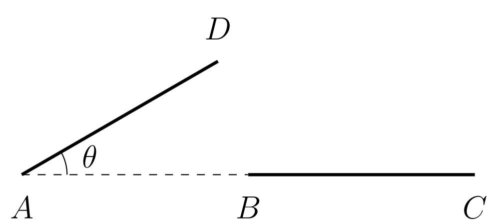
Extension: Send these waves through a speaker. Work out how to do this for waves that aren't sine waves. Does it sound like a robot, or was this just a rubbish idea?

Bonus problem (20 August)
It's a hot day and I'd like to open the windows a bit to get some airflow into the room. I've noticed that, if I open both windows a little, then the gap between the windows isn't actually that big; perhaps, counter-intuitively, I should only open one window? This geometry problem will explore that.

Here's a diagram of my windows (looking down from above).



(i) When they're closed, the windows are at AB and CB. The windows only open to an angle of θ, where $0 < \theta < 90°$, and when they're open they are at AD and CE. Find the length of DE in terms of θ; this is the gap between the windows when I open them both.



(ii) When the left window is open and the right window is closed, the windows are at AD and CB. Find the perpendicular distance from B to AD in terms of θ; this is the gap between the windows when I open one of them.

(iii) Find the range of θ for which the gap in part (i) is smaller than the gap in part (ii).

Bonus problem (27 August)
I've been watching some best-of-five esports matches. This is a series of five games between two teams, with the first team to win three games being declared the winner of the match. I don't support any of the teams in particular, but I'm interested in the different narrative arcs that you can get out of a best-of-five match. I suppose really it's a first-to-three match, because they stop playing games once one of the teams has won three of the best-of-five.

As a simpler example to show you what I mean, let's think about a best-of-three match between team A and team B. It might be the case that team A wins both the first and second games (and the third game doesn't get played). Alternatively, it might be the case that team B wins both the first and second games, but because I don't support either team in particular, I count that as the same narrative arc. The other narrative arcs are;

- one team wins the first game, the other team draws level in game two, but then the team that won the first game wins the third game
- the "reverse sweep"; one team wins the first game, but then the other team wins two in a row

Find and describe all of the possible narrative arcs for a best-of-five match.

Extension: How many possible narrative arcs are there for a best-of-$(2n + 1)$ match?

Bonus problem (03 September)
A triangle ABC has side lengths $AB = 3$, $BC = 5$, $CA = 7$. Find the angle $\angle ABC$.

Find another triangle with integer side lengths and the same angle at B.

Extension: Find another triangle with integer side lengths and half the angle at B.

For more information or to check your answers, search for "Eisenstein triple".

Bonus problem (10 September)
In some car racing games or sports, you have a sort of power-boost you can use for a short amount of time. This question will consider a driver with two different speed boosts available; should they use both at the same time or one after the other?

Suppose that the car has power P normally, and has two ways to briefly increase this. While using the first power-boost, the power is increased to $P + P_1$ for T seconds. While using the second power-boost, the power is increased to $P + P_2$ for T seconds. If both power-boosts are used simultaneously, the power is increased to $P + P_1 + P_2$ for T seconds. The velocity of the car in any case is the cube root of its power.

Should the driver use one power-boost and then immediately use the other, or would it be better to use both power-boosts together, and then drive normally for T seconds– which of these options makes the car go further over the course of $2T$ seconds?

Bonus problem (17 September)
This is a game for two players (player 1 and player 2). First each player will choose a sequence three coin flips; either heads (H) and tails (T). Then we will flip a fair coin until one of the chosen sequences happens. For example, if player 1 chooses "HTT" and player 2 chooses "THH" and then the coin flips land as "HTHTHH" then player 2 wins.

Each player must choose a sequence of three results, with player 1 choosing first, and then player 2 choosing a different sequence of three results. Show that player 2 can choose strategically based on player 1's choice, so that they have more than 50% chance to win the game.

This is called Penney's game, and the solution to the question above is on Wikipedia.

Bonus problem (24 September)
It's a remarkable fact that

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$$

You might have already seen this; if so, take a moment to think about how strange this fact is.

I'd like you to use that fact to calculate

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x$$

[Hints: expand the numerator, use polynomial division to simplify the fraction into a quotient plus a remainder over $1+x^2$, then integrate term by term.]

This integral has its own Wikipedia page. Can you see why?

I can't tell you the name of the Wikipedia page without giving the game away, but if you search Wikipedia for the phrase "requiring only elementary techniques" then it's the first result, if you want to read more about this.

Bonus bonus problem (05 November)
One of the sides of a square is part of the x-axis and the other two corners lie on the curve $y = 1 - x^2$. Find the square.

Two opposite corners of a square lie at $(0,0)$ and at $(0,1)$. The other two corners both lie on the curve $y = 1 - ax^2$ where a is a real constant. Find a.

Extension: (Difficult) Given a real number $b > 0$, is there a square with one corner at $(b, 1-b^2)$ and two other corners also on the parabola $y = 1 - x^2$?