# Vortex formation for a non-local interaction model with Newtonian repulsion and superlinear mobility

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## Abstract

We consider density solutions for gradient flow equations of the form $u_t = \nabla \cdot (\gamma(u)\nabla \mathrm{N}(u))$, where $\mathrm{N}$ is the Newtonian repulsive potential in the whole space $\mathbb{R}^d$ with the nonlinear convex mobility $\gamma(u) = u^\alpha$, and $\alpha > 1$. We show that solutions corresponding to compactly supported initial data remain compactly supported for all times leading to moving free boundaries as in the linear mobility case $\gamma(u) = u$. For linear mobility it was shown that there is a special solution in the form of a disk vortex of constant intensity in space $u = c_1 t^{-1}$ supported in a ball that spreads in time like $c_2 t^{1/d}$, thus showing a discontinuous leading front or shock. Our present results are in sharp contrast with the case of concave mobilities of the form $\gamma(u) = u^\alpha$, with $0 < \alpha < 1$ studied in [9]. There, we developed a well-posedness theory of viscosity solutions that are positive everywhere and moreover display a fat tail at infinity. Here, we also develop a well-posedness theory of viscosity solutions that in the radial case leads to a very detail analysis allowing us to show a waiting time phenomena. This is a typical behavior for nonlinear degenerate diffusion equations such as the porous medium equation. We will also construct explicit self-similar solutions exhibiting similar vortex-like behaviour characterizing the long time asymptotics of general radial solutions under certain assumptions. Convergent numerical schemes based on the viscosity solution theory are proposed analysing their rate of convergence. We complement our analytical results with numerical simulations ilustrating the proven results and showcasing some open problems.

## 1  Introduction

We are interested in the family of equations of the form

$$\begin{cases} u_t = \nabla \cdot (\gamma(u)\nabla v) & (0, +\infty) \times \mathbb{R}^d, \\ -\Delta v = u & (0, +\infty) \times \mathbb{R}^d, \\ u = u_0 & t = 0, \end{cases}$$

where the function $\gamma(u)$ is called the mobility. They all correspond to gradient flows with nonlinear mobility of the Newtonian repulsive interaction potential in dimension $d \geq 1$

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^d} \mathrm{N}(u) u \, \mathrm{d}x,$$

with $\mathrm{N}(u)$ the Newtonian repulsive potential [7], as they can be written in the form

$$\begin{cases} u_t + \nabla \cdot (\gamma(u)w) = 0 & (0, +\infty) \times \mathbb{R}^d, \\ w = -\nabla \dfrac{\delta \mathcal{F}}{\delta u} & (0, +\infty) \times \mathbb{R}^d. \end{cases} \tag{1.1}$$

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We will consider nonnegative data and solutions. The linear case $\gamma(u) = u$ is well-known in the literature as a model for wave propagation in superconductivity or superfluidity, cf. Lin and Zhang [15], Ambrosio, Mainini, and Serfaty [3, 4], Bertozzi, Laurent, and Léger [6], Serfaty and Vazquez [17]. In that case the theory leads to uniqueness of bounded weak solutions having the property of compact space support, and in particular there is a special solution in the form of a disk vortex of constant intensity in space $u = c_1 t^{-1}$ supported in a ball that spreads in time like $c_2 t^{1/d}$, thus showing a discontinuous leading front or shock. This vortex is the generic attractor for a wide class of solutions.

We want to concentrate on models with nonlinear mobility of power-like type $\gamma(u) = u^\alpha$, $\alpha > 0$. The sublinear concave $0 < \alpha < 1$ range was studied in our previous paper [9]. For nonnegative data the study provides a theory of viscosity solutions for radially symmetric initial data that are positive everywhere, and moreover display a fat tail at infinity. In particular the standard vortex of the linear mobility transforms into an explicit selfsimilar solution that reminds of the Barenblatt solution for the fast diffusion equation. A very detailed analysis is done for radially symmetric data and solutions via the corresponding mass function that satisfies a first-order Hamilton-Jacobi equation.

The present paper contains the rest of the analysis of power-like mobility for convex superlinear cases when $\gamma(u) = u^\alpha$, and $\alpha > 1$. Again, we perform a fine analysis in the case of radially symmetric solutions by means of the study of the corresponding mass function. The theory of viscosity solutions for the mass function still applies. As for qualitative properties, let us stress that in this superlinear parameter range $\alpha > 1$ solutions recover the finite propagation property and the existence of discontinuity fronts (shocks). We analyse in detail how the stable asymptotic solution goes from the fat tail profile of the sublinear case $\alpha < 1$ to the shock profile of the range $\alpha > 1$ when passing through the critical value $\alpha = 1$.

Another important aspect of the well-posedness theory that we develop for viscosity solutions with radially symmetric initial data, is that the classical approach based on optimal transport theory for equations of the form (1.1) developed in [2, 12, 7] fail for convex superlinear mobilities as described in [7] since the natural associated distance is not well-defined [12]. Therefore, our present results are the first well-posedness results for gradient flows with convex superlinear power-law mobilities, even if only for radially symmetric initial data. Finally, let us mention that we still lack of a well-posedness theory for gradient flow equations of the form (1.1) with convex superlinear mobilities for general initial data, possibly showing that the vortex-like solutions are generic attractors of the flow.

We also highlight how different convex superlinear mobilities are with respect to the linear mobility case by showing the property of an initial waiting time for the spread of the support for radial solutions that we are able to characterize. Indeed, let $u_0$ be radial and supported in a ball: $\operatorname{supp} u_0 = \overline{B}_R$. We prove that there is finite waiting time at $r = R$ if and only if

$$\limsup_{r \to R^-} \; (R - r)^{\frac{1-\alpha}{\alpha}} \int_{r < |x| < R} u_0(x)\,\mathrm{d}x = C < +\infty. \tag{1.2}$$

The waiting time phenomenon is typical of slow diffusion equations like the Porous Medium Equation [19] or the $p$-Laplacian equation. In our class of equations it does not occur for the whole range $0 < \alpha \le 1$. We are able to estimate the waiting time in terms of the limit constant $C$ in (1.2).

We combine the theory with the computational aspect: we identify suitable numerical methods and perform a detailed numerical analysis. Indeed, we construct numerical finite-difference convergent schemes and prove convergence to the actual viscosity solution for radially symmetric solutions based on the mass equation. By taking advantage of the connection to nonlinear Hamilton-Jacobi equations, we obtain monotone numerical schemes showing their convergence to the viscosity solutions of the problem with a uniform rate of convergence, see Theorem 5.6 in constrast with the case of concave sublinear mobilities in [9].

The paper is structured as follows. We start by constructing some explicit solutions and developing the theory for radially symmetric initial data by using the mass equation in Sections 2 and 3 respectively. We construct the general viscosity solution theory for radially symmetric initial data

in Section 4 showing the most striking phenomena for convex superlinear mobilities: compactly supported free boundaries determined by sharp fronts and the waiting time phenomena. Section 5 is devoted to show the convergence of monotone schemes for the developed viscosity solution theory with an explicit convergence rate. The numerical schemes constructed illustrate the sharpness of the waiting time phenomena result, and allow us to showcase interesting open problems in Section 6. A selection of figures illustrates a number of salient phenomena. We provide videos for some interesting situations as supplementary material in [20].

## 2 Explicit solutions

The aim of this section is to find some important explicit solutions of

$$\begin{cases} u_t = \nabla \cdot (u^\alpha \nabla v) & (0, +\infty) \times \mathbb{R}^d \\ -\Delta v = u & (0, +\infty) \times \mathbb{R}^d \\ u = u_0 & t = 0 \end{cases} \tag{P}$$

Notice that, as in [9], we still have that, for $C \geq 0$

$$\overline{u}(t) = (C + \alpha t)^{-\frac{1}{\alpha}} \tag{2.1}$$

is a solution of the PDE. The repulsion potential $v$ diverges quadratically at infinity. For $C = 0$ we recover the Friendly Giant solution, introduced in [9].

### 2.1 Self-similar solution

The algebraic calculations developed in [9] still work, we get self-similar solutions of the form

$$U(t, x) = t^{-\frac{1}{\alpha}} F(|x| t^{-\frac{1}{\alpha d}}),$$

with the self-similar profile

$$F(|y|) = \left( \alpha + \left( \frac{\omega_d |y|^d}{\alpha} \right)^{-\frac{\alpha}{\alpha-1}} \right)^{-\frac{1}{\alpha}}. \tag{2.2}$$

Let us remark that, for $\alpha > 1$, we have $F(0) = 0$ and $F(+\infty) = \alpha^{-\frac{1}{\alpha}}$ (see Figure 1 for a sketch of the self-similar profiles depending on $\alpha$). This is different to the case $0 < \alpha < 1$, where $F(0)$ is a positive constant and $F$ decays at infinity. In our present case $\alpha > 1$, the self-similar solutions have infinite mass, whereas for $0 < \alpha < 1$ the self-similar solutions have finite mass.
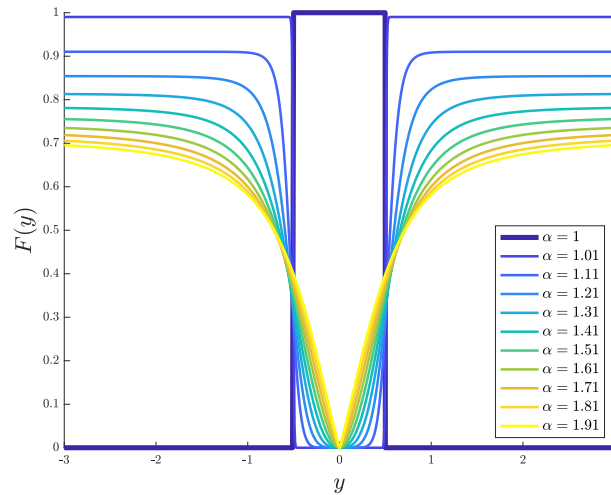


Figure 1: Self-similar profiles when $d = 1$

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For $0 < \alpha < 1$ these self-similar give the typical asymptotic behaviour as $t \to +\infty$. For $\alpha > 1$ we will show this is no longer the case, for finite mass solutions.

## 2.2  Vortexes

The vortex solutions defined as

$$u(t,x) = \begin{cases} (u_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} & \omega_d |x|^d < S(t) = M/(u_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

are local weak solutions of (P).

**Remark 2.1.** Notice that, for $t \to -u_0^{-\alpha}/\alpha$, the vortex collapses to the Dirac delta of mass $M$.

This solution was also constructed by characteristics and the Rankine-Hugoniot condition in [9, Section 5.2]. However, in that case the Lax-Oleinik condition of incoming characteristics failed. In our present setting for $\alpha > 1$, this shock-type solutions are entropic, and we will prove that they are indeed viscosity solutions of the mass equation (3.1). We will prove that, for $\alpha > 1$, they now have significant relevance. In particular, they describe the asymptotic behaviour as $t \to +\infty$. Notice that the the radius of the support $S(t)$ of this kind of solutions solves an equation of type

$$\frac{\mathrm{d}S}{\mathrm{d}t} = M(u_0^{-\alpha} + \alpha t)^{-1+\frac{1}{\alpha}}. \tag{2.4}$$

There are also complementary vortex solutions:

$$u(t,x) = \begin{cases} 0 & |x|^d < a \\ (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} & |x|^d > a \end{cases} \tag{2.5}$$

which are stationary (and solve the mass problem (3.1) by characteristics). This type of solution belongs to a theory of solutions in $L^\infty$, but not $L^1$.

# 3  Mass of radial solutions

In [9] we proved that the mass of a radial solution

$$m(t,r) = \int_{B_r} u(t,x)\,\mathrm{d}x$$

written in volume coordinates $\rho = d\omega_d r^d$ is a solution of the equation

$$m_t + m(m_\rho)^\alpha = 0. \tag{3.1}$$

## 3.1  Characteristics for the mass equation

In [9] we computed the generalised characteristics for the mass equation in the case of sublinear mobility, $\alpha < 1$. The algebra for characteristics still works

$$\rho = \rho_0 + \alpha m(\rho_0) u_0(\rho_0)^{\alpha-1} t, \tag{3.2}$$

and the solutions behave like

$$u(t,\rho) = (u_0(\rho_0)^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}, \tag{3.3}$$

and

$$m(t,\rho) = m_0(\rho_0)\left(1 + \alpha u_0(\rho_0)^\alpha t\right)^{1-\frac{1}{\alpha}}.$$

**Remark 3.1.** Notice that the the generalised characteristics are not the level sets of $m$.

These solutions are well defined, for a given $u_0$, while the characteristics do not cross.

**Proposition 3.2.** *Let $u_0$ be non-decreasing and $C^1$. Then, there is a classical global solution of the mass equation, given by characteristics.*

*Proof.* Let $P_t(\rho_0) = \rho_0 + \alpha m(\rho_0)u_0(\rho_0)^{\alpha-1}t$. Clearly $P_t(0) = 0$. If $u_0$ is non-decreasing, $\frac{dP_t}{d\rho_0} \geq 1$, and hence it is invertible. We construct

$$u(t,\rho) = \begin{cases} \left(\left(u_0(P_t^{-1}(\rho))\right)^{-\alpha} + \alpha t\right)^{-\frac{1}{\alpha}} & \text{if } u_0(P_t^{-1}(\rho)) \neq 0, \\ 0 & \text{if } u_0(P_t^{-1}(\rho)) = 0. \end{cases} \tag{3.4}$$

It is immediate to see that $u$ is continuous and $C^1$. $\qquad\qquad\square$

For $0 < \alpha < 1$, in [9] we developed a theory of classical solutions for non-increasing initial data. In that case, rarefaction fan tails appeared, which gave rise to classical solutions of the mass equation. In our present case $\alpha > 1$, it seems that the good data is radially non-decreasing, but this is not possible in an $L^1 \cap L^\infty$ theory, unless a jump is introduced.

## 3.2 The Rankine-Hugoniot condition

We will prove in Section 4.3 that solutions with a jump, given by a Rankine-Hugoniot condition, will be the correct "stable" solutions. As in [9], shocks (i.e. discontinuities) propagate following a Rankine-Hugoniot condition. If $S(t)$ is the position of the shock, we write the continuity of mass condition

$$m(t, S(t)^-) = m(t, S(t)^+).$$

Taking a derivative and applying the equation (3.1) we have that

$$\frac{\mathrm{d}S}{\mathrm{d}t}(t) = m(t, S(t))\frac{u(t, S(t)^+)^\alpha - u(t, S(t)^-)^\alpha}{u(t, S(t)^+) - u(t, S(t)^-)}. \tag{3.5}$$

**Remark 3.3.** Notice that, in the case of the vortex the Rankine-Hugoniot condition determines precisely the support. In particular, we have $u(t, S(t)^+) = 0$ and $u(t, S(t)^-) = (u_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}$ so (3.5) is precisely (2.4). In fact, the vortex is simply a cut-off of the Friendly Giant with a free-boundary determined by the Rankine-Hugoniot condition.

## 3.3 Local existence of solutions by characteristics

**Theorem 3.4.** *Let $0 \leq u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be radial and such that $u_0^{\alpha-1}$ is Lipschitz. Then, there exists a small time $T > 0$ and a classical solution of the mass equation given by characteristics defined for $t \in [0, T]$.*

*Proof.* The solution given by (3.2) and (3.3) is well defined as long as the characteristics cover the whole space, and do not cross. This is equivalent to $P_t(\rho_0) = \rho_0 + \alpha m(\rho_0)u_0(\rho_0)^{\alpha-1}t$ being a bijection $[0, +\infty) \to [0, +\infty)$. Again, we construct (3.4). Since $P_t(0) = 0$, it suffices to prove that $\frac{dP_t}{d\rho_0} \geq c_0 > 0$. We take the derivative explicitly and find that

$$\begin{aligned} \frac{\mathrm{d}P_t}{\mathrm{d}\rho_0} &= 1 + \alpha t\left(\frac{\mathrm{d}m_0}{\mathrm{d}\rho_0}u_0^{\alpha-1} + m_0\frac{\mathrm{d}}{\mathrm{d}\rho_0}(u_0^{\alpha-1})\right) \\ &= 1 + \alpha t\left(u_0^\alpha + m_0\frac{\mathrm{d}}{\mathrm{d}\rho_0}(u_0^{\alpha-1})\right). \end{aligned}$$

Due to the hypothesis

$$L := \sup_{\rho_0 \geq 0} \left| u_0^\alpha + m_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}(u_0^{\alpha-1}) \right| < \infty,$$

and we have that $\frac{\mathrm{d}P_t}{\mathrm{d}\rho_0} \geq 1 - \alpha t L$ which is strictly positive if $t \leq T < 1/(\alpha L)$. Since $u_0^{\alpha-1}$ is Lipschitz and bounded, then $1 - \alpha T L \leq \frac{\mathrm{d}P_t}{\mathrm{d}\rho_0} \leq C$ is Lipschitz. Hence, $P_t^{-1}$ is Lipschitz in $\rho_0$. Also, it is easy to see that $P_t^{-1}$ is continuous in $t$. Since it is immediate to check that $u$ is continuous by composition, we have that $m$ is of class $C^1$, and the proof is complete. □

**Corollary 3.5** (Waiting time). *Let $0 \leq u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be radial and such that $u_0^{\alpha-1}$ is Lipschitz. Then, there is a short time $T > 0$ such that, if $\operatorname{supp} u_0 \subset \overline{B_R}$, then any classical solution of the mass equation satisfies $\operatorname{supp} u(t, \cdot) \subset \overline{B_R}$ for $t < T$.*

*Proof.* Notice that, if $u_0$ is compactly supported, then outside the support the characteristics are given by $P_t(\rho_0) = \rho_0$. As long as the solution is given by characteristics, if $\operatorname{supp} u_0 \subset \overline{B_R}$ for $\rho > \omega_d R^d$, then $u(t, \rho) = 0$. □

**Remark 3.6.** This effect of preservation of the support for a finite time is known as *waiting time*. In Section 3.4 we will show this holds true as long as the solution is smooth. In Section 4.5 we show that this waiting time effect must be finite. This will lead us to show that solutions must loose regularity.

**Remark 3.7.** Notice that higher regularity of $u_0$ is preserved by characteristics. Taking a derivative

$$\frac{\mathrm{d}u}{\mathrm{d}\rho} = \left( \left(u_0(P_t^{-1}(\rho))\right)^{-\alpha} + \alpha t \right)^{-1-\frac{1}{\alpha}} u_0^{-1-\alpha} \frac{\mathrm{d}u_0}{\mathrm{d}\rho_0} \frac{\mathrm{d}P_t^{-1}}{\mathrm{d}\rho.}$$

$$= \left(1 + \alpha t u_0(P_t^{-1}(\rho))^\alpha\right)^{-1-\frac{1}{\alpha}} \frac{\mathrm{d}u_0}{\mathrm{d}\rho_0} \frac{1}{\frac{\mathrm{d}P_t}{\mathrm{d}\rho_0}}$$

$$= \frac{\left(1 + \alpha t u_0(P_t^{-1}(\rho))\right)^\alpha\right)^{-1-\frac{1}{\alpha}}}{1 + \alpha t \left(u_0^\alpha + m_0 \frac{\mathrm{d}}{\mathrm{d}\rho_0}(u_0^{\alpha-1})\right)} \frac{\mathrm{d}u_0}{\mathrm{d}\rho_0}$$

It is easy to see that, for small time, if $u_0$ is smooth enough, then $u$ is of class $\mathcal{C}^1$.

**Remark 3.8.** The condition $u_0^{\alpha-1}$ Lipschitz is sharp. Let us take, for $\varepsilon > 0$

$$u_0(\rho) = (c_0 - \rho)_+^{\frac{1-\varepsilon}{\alpha-1}} \tag{3.6}$$

and let us show that characteristics cross for all $t > 0$. Looking at the characteristics for $\rho_0 = c_0 - \delta$ with $\delta$ positive but small (so that $m_0(\rho) > M/2$) we have that

$$\rho = \rho_0 + \alpha m_0(\rho)(c_0 - \rho_0)_+^{1-\varepsilon} t$$

$$\geq c_0 - \delta + \alpha \frac{M}{2} \delta^{1-\varepsilon} t$$

But for $\delta < \left(\frac{\alpha M t}{2}\right)^{\frac{1}{\varepsilon}}$ we have $\rho > c_0$. But this is not possible, since it must have crossed the characteristic $\rho = c_0$ coming from $\rho_0 = c_0$. No solutions by characteristics can exist.

The crossing of characteristics will immediately lead to the formation of shock waves. The shock waves will be lead by a Rankine-Hugoniot condition as above.

**Remark 3.9.** As shown in Remark 3.8, with initial datum (3.6) we cannot expect solutions by characteristics. We could potentially paste solutions by characteristics on either side of a shock. We will show that this is the case, and we will show that solutions with bounded and compactly supported initial data will indeed produce a propagating shock at the end of their support, possibly with a waiting time (see the main results in Section 4).

## 3.4 Explicit Ansatz with waiting time

For fixed mass $M$ and prescribed support of $u = m_\rho$ we can construct local classical solutions with waiting time $T$. We will prove that

$$m(t,\rho) = \left( M^{\frac{\alpha}{\alpha-1}} - \alpha^{\frac{1}{\alpha-1}} \frac{(c_0 - \rho)_+^{\frac{\alpha}{\alpha-1}}}{(T-t)^{\frac{1}{\alpha-1}}} \right)^{\frac{\alpha-1}{\alpha}}, \qquad \text{if } t < T \text{ and } \rho > \left( c_0 - \alpha^{\frac{1}{\alpha}} M(T-t)^{\frac{1}{\alpha}} \right)_+ \qquad (3.7)$$

is a classical solution of $m_t + m_\rho^\alpha m = 0$. We represent this function in Figure 2. We will extend this function by zero $\rho \le \left( c_0 - \alpha^{\frac{1}{\alpha}} M(T-t)^{\frac{1}{\alpha}} \right)_+$ to construct a viscosity subsolution of the mass equation.
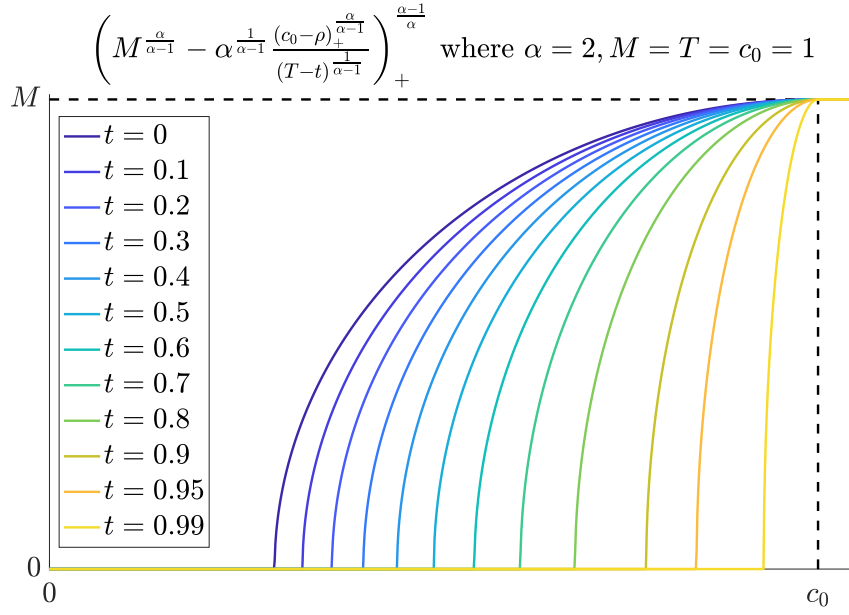
$$\left( M^{\frac{\alpha}{\alpha-1}} - \alpha^{\frac{1}{\alpha-1}} \frac{(c_0-\rho)_+^{\frac{\alpha}{\alpha-1}}}{(T-t)^{\frac{1}{\alpha-1}}} \right)^{\frac{\alpha-1}{\alpha}}_+ \quad \text{where } \alpha = 2, M = T = c_0 = 1$$



Figure 2: Ansatz solution

The intuition to construct this kind of explicit Ansatz in "separated variables" is well known in the context of nonlinear PDEs of power type (see, e.g., [19]). One starts with a general formula of the type

$$m(t,\rho) = \left( M^{\frac{1}{\beta}} - c \frac{(c_0 - \rho)^\gamma}{(T-t)^\delta} \right)^\beta$$

and we match the exponents through the scaling properties of the equation. By taking $\beta = \frac{\alpha-1}{\alpha}$, $\gamma = \frac{\alpha}{\alpha-1} > 1$ and $\delta = \frac{1}{\alpha-1}$ this gives

$$m_t + m m_\rho^\alpha = \left( M^{\frac{1}{\beta}} - c \frac{(c_0 - \rho)^\gamma}{(T-t)^\delta} \right)^{\beta-1} \frac{(c_0 - \rho)^\gamma}{(T-t)^{\delta+1}} \left[ -c\delta\beta + (c\gamma\beta)^\alpha \right].$$

Hence, the sign is that of $-c\delta\beta + (c\gamma\beta)^\alpha$, which in our case is precisely $-\frac{c}{\alpha} + c^\alpha$.

**Remark 3.10.** We have also checked the following properties

1. $m \in \mathcal{C}^1$ in its domain of definition, since $\gamma \ge 1$, the matching at $\rho = c_0$ is guaranteed by the explicit formula for $m_\rho$.

2. The domain of definition of $m$ depends on the value of $T$, and $\rho = 0$ may not the contained in the domain.

3. It is easy to check that the function $u = m_\rho$ associate to the Ansatz satisfies

$$u(t, \rho) = \frac{c}{m(t, \rho)^{\frac{1}{\alpha-1}} (T - t)^{\frac{1}{\alpha-1}}} (c_0 - \rho)_+^{\frac{1}{\alpha-1}}. \qquad (3.8)$$

Notice that $u^{\alpha-1}$ is a Lipschitz function of $\rho$. The easiest way to check this expression is by using

$$m^{\frac{\alpha}{\alpha-1}} = M^{\frac{\alpha}{\alpha-1}} - c \frac{(c_0 - \rho)_+^{\frac{\alpha}{\alpha-1}}}{(T - t)^{\frac{1}{\alpha-1}}},$$

taking a derivative in $\rho$, and solving for $m_\rho$.

4. Conversely, notice that the condition $u_0^{\alpha-1}$ in Theorem 3.4 is sharply satisfied by

$$u_0(\rho) \sim (c_0 - \rho)_+^{\frac{1}{\alpha-1}}.$$

If this is the behaviour of $u_0$ then close to $\rho = c_0$ we have that

$$(m_0^{\frac{\alpha}{\alpha-1}})_\rho = \frac{\alpha}{\alpha-1} m_0^{\frac{1}{\alpha-1}} (m_0)_\rho = \frac{\alpha}{\alpha-1} m^{\frac{1}{\alpha-1}} u_0 \sim \frac{\alpha}{\alpha-1} M^{\frac{1}{\alpha-1}} (c_0 - \rho)_+^{\frac{1}{\alpha-1}}$$

and, integrating in $[\rho, c_0]$

$$M^{\frac{\alpha}{\alpha-1}} - m_0^{\frac{\alpha}{\alpha-1}} \sim M^{\frac{1}{\alpha-1}} (c_0 - \rho)_+^{\frac{\alpha}{\alpha-1}}.$$

Solving for $m$, we precisely recover the Ansatz

$$m_0 \sim \left( M^{\frac{\alpha}{\alpha-1}} - M^{\frac{1}{\alpha-1}} (c_0 - \rho)_+^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}}.$$

## 3.5  A change of variable to a Hamilton-Jacobi equation

The change in variable $m = \theta^{\frac{\alpha-1}{\alpha}}$ leads to the equation

$$\theta_t + \left( \frac{\alpha-1}{\alpha} \right)^{\alpha-1} \theta_\rho^\alpha = 0. \qquad (3.9)$$

This equation is of classical Hamilton-Jacobi form, and falls in the class studied by Crandall and Lions in the famous series of papers (see, e.g. [11]). Letting $w = \theta_\rho$ we recover a Burguer's conservation equation

$$w_t + \left( \frac{\alpha-1}{\alpha} \right)^{\alpha-1} (w^\alpha)_\rho = 0. \qquad (3.10)$$

The theory of existence and uniqueness of entropy solutions for this problem is well-know (see, e.g. [5, 8, 14] and related results in [13, 16]). Notice that the relation between $u$ and $w$ is somewhat difficult

$$w(t, \rho) = \frac{\mathrm{d}}{\mathrm{d}\rho} \left[ \left( \int_0^\rho u(t, \sigma) \, \mathrm{d}\sigma \right)^{\frac{\alpha-1}{\alpha}} \right].$$

**Remark 3.11.** Notice that, for $\alpha < 1$ this change of variable does not make any sense since as $\rho \to 0$ we have that $m \to 0$ so $\theta \to +\infty$. We would be outside the $L^1 \cap L^\infty$ framework.

# 4  Viscosity solutions of the mass equation

## 4.1  Existence, uniqueness and comparison principle

The Crandall-Lions theory of viscosity solutions developed in [9] for $\alpha < 1$, works also for the case $\alpha \geq 1$ without any modifications. Since $m_\rho = u \geq 0$ we can write the equation as

$$m_t + (m_\rho)_+^\alpha m = 0. \qquad (4.1)$$

Then, the Hamiltonian $H(z, p_1, p_2) = (p_2)^\alpha z$ is defined and non-decreasing everywhere. We write the initial and boundary conditions

$$\begin{cases} m_t + (m_\rho)_+^\alpha m = 0 & t, \rho > 0 \\ m(t, 0) = 0 & t > 0 \\ m(0, \rho) = m_0(\rho) & \rho > 0. \end{cases} \tag{4.2}$$

The natural setting is with $m_0$ Lipschitz (i.e. $m_\rho = u \in L^\infty$) and bounded (i.e. $u \in L^1$). We introduce the definition of viscosity solution for our problem and some notation.

**Definition 4.1.** Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}$. We define the Fréchet subdifferential and superdifferential

$$D^- u(x) = \left\{ p \in \mathbb{R}^m : \liminf_{y \to x} \frac{u(y) - u(x) - p(y - x)}{|y - x|} \geq 0 \right\}$$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^m : \limsup_{y \to x} \frac{u(y) - u(x) - p(y - x)}{|y - x|} \leq 0 \right\}.$$

**Definition 4.2.** We say that a continuous function $m \in \mathcal{C}([0, +\infty)^2)$ is a:

- viscosity subsolution of (4.2) if

$$p_1 + (p_2)_+^\alpha m(t, \rho) \leq 0, \qquad \forall (t, \rho) \in \mathbb{R}_+^2 \text{ and } (p_1, p_2) \in D^+ m(t, \rho). \tag{4.3}$$

and $m(0, \rho) \leq m_0(\rho)$ and $m(t, 0) \leq 0$.

- a viscosity supersolution of (4.2) if

$$p_1 + (p_2)_+^\alpha m(t, \rho) \geq 0, \qquad \forall (t, \rho) \in \mathbb{R}_+^2 \text{ and } (p_1, p_2) \in D^- m(t, \rho).$$

and $m(0, \rho) \geq m_0(\rho)$ and $m(t, 0) \geq 0$.

- a viscosity solution of (4.2) if it is both a sub and supersolution.

The main results in [9] show the comparison principle and the well-posedness result for viscosity solutions of (4.2).

**Theorem 4.3.** *Let $m$ and $M$ be uniformly continuous sub and supersolution of* (4.2) *in the sense of Definition 4.2. Then $m \leq M$.*

We will denote by $\mathrm{BUC}$ the space of bounded uniformly continuous functions.

**Theorem 4.4.** *If $m_0 \in \mathrm{BUC}([0, +\infty))$ be non-decreasing such that $m_0(0) = 0$. Then, there exists a unique bounded and uniformly continuous viscosity solution. Furthermore, we have that*

$$\|m(t, \cdot)\|_\infty = \lim_{\rho \to +\infty} m(t, \rho) = \|m_0\|_\infty, \qquad \|m_\rho(t, \cdot)\|_\infty \leq \|(m_0)_\rho\|_\infty. \tag{4.4}$$

*If $m_0$ is Lipschitz, then so is $m$.*

## 4.2 The vortex

We next show that the mass associated to vortexes (2.3) (see Figure 3) are viscosity solutions if and only if $\alpha \geq 1$. In [9] we showed that the mass associated to vortex solutions are not of viscosity type for $\alpha < 1$. Intuitively, this is another instance of degenerate diffusion versus fast diffusion-like behaviour, i.e. compactly supported versus fat tails.

**Theorem 4.5.** *The mass associated to the vortex solution* (2.3)

$$m(t, \rho) = \min\{(c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} \rho, c_0 L\}. \tag{4.5}$$

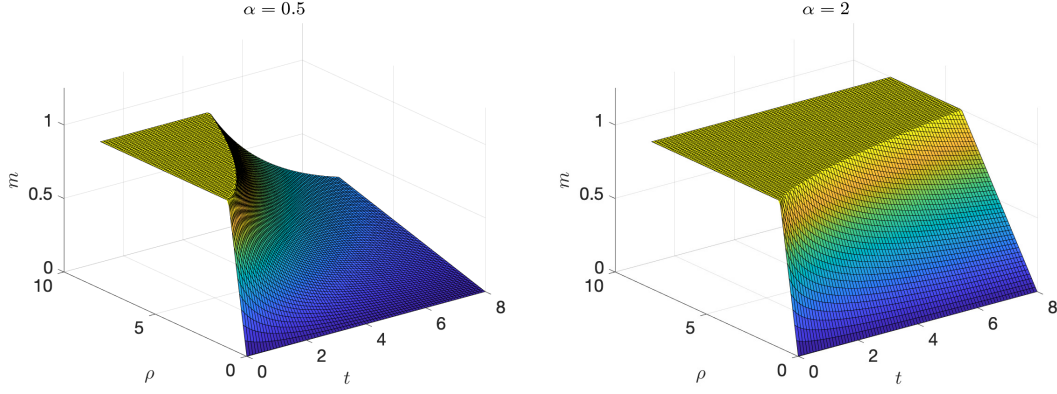*is a viscosity solution for $\alpha \geq 1$.*

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Figure 3: Representation of (4.5) where $c_0 = L = 1$

*Proof.* Let us fix a $t_0, \rho_0 > 0$. If we are not on the edge, i.e. $(c_0^{-\alpha} + \alpha t_0)^{-\frac{1}{\alpha}} \rho_0 \neq c_0 L$, then $m$ is a classical solution and it satisfies the viscosity formulation.

Let us, therefore, look at a point such that

$$(c_0^{-\alpha} + \alpha t_0)^{-\frac{1}{\alpha}} \rho_0 = c_0 L. \tag{4.6}$$

It is clear that no smooth function $\varphi \leq m$ can be tangent to $m$ at $(t_0, \rho_0)$, since the derivative of $m_\rho(t_0, \rho_0^-) > m_\rho(t_0, \rho_0^+)$. Therefore, the definition of viscosity supersolution is immediately satisfied, since $D^- m(t_0, \rho_0) = \emptyset$.

To check that $m$ is a viscosity subsolution, let us take $(p_1, p_2) \in D^+ m(t_0, \rho_0)$. Due to the explicit formula $p_2 \in [0, (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}]$, and since it is non-increasing in $t$, $p_1 \leq 0$.

If $p_2 = 0$ then, since $p_1 \leq 0$, (4.3) is satisfied. Assume $p_2 \neq 0$. There exists $\varphi \in \mathcal{C}^1$ such that $\varphi \geq m$, $\varphi(t_0, \rho_0) = m(t_0, \rho_0)$ and $\varphi_t(t_0, \rho_0) = p_1$ and

$$\varphi_\rho(t_0, \rho_0) = p_2 \leq (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}. \tag{4.7}$$

Let us take a look at the level sets. Since $\varphi \geq m$, we have that $\{\varphi < c_0 L\} \subset \{m < c_0 L\}$. Therefore $\partial\{\varphi < c_0 L\}$, is contained in the region $\overline{\{m < c_0 L\}}$. Since $\varphi$ and $m$ coincide at $(t_0, \rho_0)$, then $\partial\{\varphi < c_0 L\}$ and $\partial\{m < c_0 L\}$ are tangent at $(t_0, \rho_0)$ (see Figure 4).
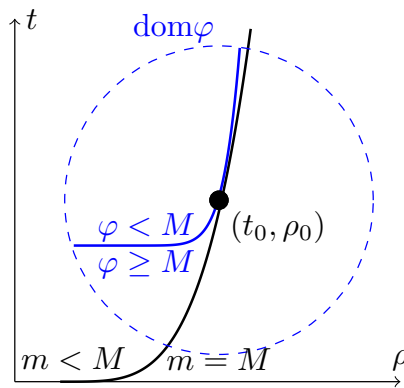


Figure 4: Relation between level sets. In the figure, $M = c_0 L$.

Since $p_2 > 0$, $\partial\{\varphi < c_0 L\}$ can be parametrised by a curve $(t, \rho^*(t))$ defined for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\varphi(t, \rho^*(t)) = c_0 L$ and $\rho^*(t_0) = \rho_0$. Hence, taking a derivative and evaluating at $(t_0, \rho_0)$

$$\varphi_t(t_0, \rho_0) + \frac{\mathrm{d}\rho^*}{\mathrm{d}t}(t_0)\varphi_\rho(t_0, \rho_0) = 0.$$

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On the other hand, $\partial\{m < c_0 L\}$ can be parametrised as $(t, S(t))$ where

$$S(t) = c_0 L (c_0^{-\alpha} + \alpha t)^{\frac{1}{\alpha}}$$

and hence

$$\frac{\mathrm{d}S}{\mathrm{d}t} = c_0 L (c_0^{-\alpha} + \alpha t)^{-1+\frac{1}{\alpha}}. \tag{4.8}$$

Notice that the sign of $\alpha - 1$ decides the convexity or concavity of the matching curve. Since the level sets are tangent at $(t_0, \rho_0)$, the derivatives coincide and we have

$$\frac{\mathrm{d}\rho^*}{\mathrm{d}t}(t_0) = \frac{\mathrm{d}S}{\mathrm{d}t}(t_0) = c_0 L (c_0^{-\alpha} + \alpha t_0)^{-1+\frac{1}{\alpha}}.$$

Hence, we have that

$$p_1 = \varphi_t(t_0, \rho_0) = -\frac{\mathrm{d}\rho^*}{\mathrm{d}t}(t_0)\varphi_\rho(t_0, \rho_0) = -c_0 L (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} p_2.$$

Applying (4.7) and that $\alpha \geq 1$ we have that

$$p_1 + m(t_0, \rho_0)p_2^\alpha = -c_0 L (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} p_2 + c_0 L p_2^\alpha$$

$$= c_0 L p_2 \left( p_2^{\alpha-1} - (c_0^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} \right)$$

$$\leq 0,$$

which is precisely (4.3). This completes the proof. $\qquad\square$

**Remark 4.6.** The notion of viscosity solution can be extended, by approximation, to cover non-negative finite measures as initial data $u_0$. Notice that these vortex solutions concentrate as $t \searrow -c_0^{-\alpha}/\alpha$ to the Heaviside function $m(t, \rho) \to c_0 L H_0(\rho)$. As we point out above, this proves that if

$$u_0 = M\delta_0 \quad \text{(i.e. } m_0 = MH_0\text{)}, \tag{4.9}$$

then the solution is a cut-off of the Friendly Giant (2.1)

$$u(t, \rho) = \begin{cases} (\alpha t)^{-\frac{1}{\alpha}} & \rho < M(\alpha t)^{\frac{1}{\alpha}} \\ 0 & \rho > M(\alpha t)^{\frac{1}{\alpha}} \end{cases},$$

and hence

$$m(t, \rho) = \begin{cases} (\alpha t)^{-\frac{1}{\alpha}}\rho & \rho \leq M(\alpha t)^{\frac{1}{\alpha}} \\ M & \rho > M(\alpha t)^{\frac{1}{\alpha}} \end{cases}. \tag{4.10}$$

For every $\varepsilon > 0$ this is a viscosity solution of (4.2) in $\mathcal{C}((\varepsilon, +\infty) \times \mathbb{R}_+)$. Notice that $m$ is of self-similar form

$$m(t, \rho) = MG\left(\frac{\rho}{M(\alpha t)^{\frac{1}{\alpha}}}\right), \qquad \text{where } G(y) = \begin{cases} y & y \leq 1 \\ 1 & y > 1 \end{cases}. \tag{4.11}$$

In fact, by translation invariance, it is possible to show that $u_0 = M\delta_{c_0}$, i.e.

$$m_0(\rho) = MH_{c_0}(\rho)$$

then

$$m(t, \rho) = \begin{cases} 0 & \rho < c_0 \\ (\alpha t)^{-\frac{1}{\alpha}}(\rho - c_0) & \rho \in [c_0, c_0 + M(\alpha t)^{\frac{1}{\alpha}}) \\ M & \rho > c_0 + M(\alpha t)^{\frac{1}{\alpha}} \end{cases}. \tag{4.12}$$

is a viscosity solution of the mass equation defined for $t > 0$. With self-similar form

$$m(t, \rho) = MG\left(\frac{\rho - c_0}{M(\alpha t)^{\frac{1}{\alpha}}}\right), \qquad \text{where } G(y) = \begin{cases} 0 & y \leq 0 \\ y & y \in (0, 1) \\ 1 & y > 1 \end{cases}. \tag{4.13}$$

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## 4.3 Monotone non-decreasing data with final cut-off

As in [9], the theory of existence and uniqueness is written in terms of $m$, but we take advantage of the intuition from the conservation law (P) for $u$, to construct explicit solutions through characteristics. Notice that taking a derivative in (4.2) we can write

$$u_t + (u^\alpha m)_\rho = 0.$$

Afterwards, we check that the constructed solution, fall into our viscosity theory for $m$.

Since it is suggested by (3.2) that characteristics do not cross if $u_0$ is non-decreasing, let us look for solutions with initial data

$$u_0(\rho) = \begin{cases} \widetilde{u}_0(\rho) & \rho < S_0 \\ 0 & \rho > S_0. \end{cases}, \qquad \text{where } \widetilde{u}_0 \text{ in continuous and non-decreasing in } [0, S_0]. \qquad (4.14)$$

When $m_0$ and $u_0$ are non-decreasing, it clear that characteristics with foot on $\rho_0 \in [0, S_0]$ do not cross. If $\widetilde{u}_0 \not\equiv 0$, then $u_0(S_0^-) = \widetilde{u}_0(S_0) > 0$ and there is a shock starting at $S_0$ which will propagate following the Rankine-Hugoniot condition (3.5).

We construct the viscosity solution as follows. The characteristic of foot $\rho_0 = S_0$ is precisely

$$\overline{S}(t) = S_0 + \alpha M \widetilde{u}_0(S_0)^{\alpha-1} t.$$

For $\rho < \overline{S}(t)$ we can go back through the characteristics with $P_t$ defined above. Let us define

$$\widetilde{u}(t, \rho) = \begin{cases} 0 & \text{if } \widetilde{u}_0(\rho) = 0 \\ (\widetilde{u}_0(P_t(\rho))^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}} & \text{if } \widetilde{u}_0(\rho) > 0 \text{ and } \rho < \overline{S}(t) , \\ 0 & \text{if } \rho > \overline{S}(t) \end{cases} \qquad (4.15)$$

The shock is given by

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = M \widetilde{u}(t, S(t))^{\alpha-1} \\ S(0) = S_0 \end{cases}. \qquad (4.16)$$

Finally we define

$$u(t, \rho) = \begin{cases} \widetilde{u}(t, \rho) & \rho < S(t) \\ 0 & \rho > S(t). \end{cases} \qquad (4.17)$$

Solving explicitly is not possible. However, we can prove that

**Proposition 4.7.** *Let $\alpha \geq 1$, (4.14). Then, the mass of (4.17) is a viscosity solution of (4.2) and $S(t) \leq \overline{S}(t)$.*

*Proof.* Since $\alpha \geq 1$, we have that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = m(t, S(t)) u(t, S(t)^-)^{\alpha-1} = m(t, S(t)) \widetilde{u}(t, S(t))^{\alpha-1} \leq \alpha M \widetilde{u}_0(S_0)^{\alpha-1} = \frac{\mathrm{d}\overline{\rho}}{\mathrm{d}t}.$$

where $M = \sup m_0$. Also $S(0) = \overline{S}(0)$. Hence the shock is slower than the last characteristic (i.e. $S(t) \leq \overline{S}(t)$) and this implies that there are no outgoing characteristics and the Lax-Oleinik condition is satisfied.

Now, in order to check that it is a viscosity solution, we can repeat the proof of Theorem 4.5, replacing (4.8) by (4.16). $\qquad \square$

## 4.4 Two Dirac deltas

Let us now consider that

$$u_0 = m_1 \delta_{\rho_1} + m_2 \delta_{\rho_2}. \tag{4.18}$$

where $0 \leq \rho_1 < \rho_2$. Then the initial mass $m_0$ is discontinuous, and this creates some technical difficulties. We will show the viscosity solution for (4.2) is the primitive of

$$u(t,\rho) = \begin{cases} 0 & \rho < \rho_1 \\ (\alpha t)^{-\frac{1}{\alpha}} & \rho \in [\rho_1, S_1(t)] \\ 0 & \rho \in [S_1(t), \rho_2] \\ \left( \left( \frac{\rho - \rho_2}{\alpha m_1 t} \right)^{-\frac{\alpha}{\alpha-1}} + \alpha t \right)^{-\frac{1}{\alpha}} & \rho \in [\rho_2, S_2(t)] \\ 0 & \rho > S_2(t) \end{cases} \tag{4.19}$$

where

$$S_1(t) = \rho_1 + m_1 (\alpha t)^{\frac{\alpha-1}{\alpha}}, \tag{4.20}$$

and

$$S_2(t) = \rho_2 + \alpha m_1 \mathcal{K}^{-1} \left( \frac{m_2}{\alpha m_1} \right) t^{\frac{1}{\alpha}}, \tag{4.21}$$

with

$$\mathcal{K}(\tau) = \int_0^\tau \left( s^{-\frac{\alpha}{\alpha-1}} + \alpha \right)^{-\frac{1}{\alpha}} \mathrm{d}s. \tag{4.22}$$

We have the following estimates for the function $\mathcal{K}^{-1}$: for $\tau \leq s_0$ there exists $c(s_0), C(s_0) > 0$ such that

$$c(s_0)\tau^{\frac{\alpha}{\alpha-1}} \leq \mathcal{K}^{-1}(\tau) \leq C(s_0)\tau^{\frac{\alpha}{\alpha-1}}. \tag{4.23}$$

This solution is defined for all $t$ such that $S_1(t) \leq \rho_2$, i.e. $t \leq ((\rho_2 - \rho_1)/m_1)^{\frac{\alpha}{\alpha-1}}/\alpha$. For large $t$, $S_1$ would need to be computed from another further Rankine-Hugoniot condition. We will only use the value for $t$ small, so this computation is enough for our purposes.
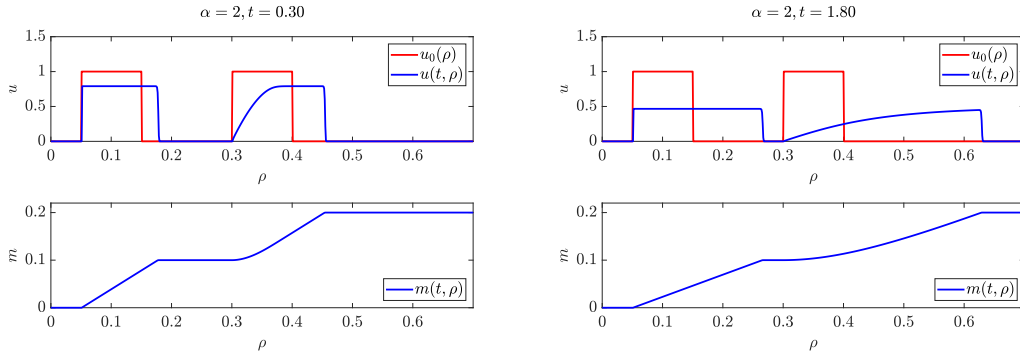


Figure 5: Solutions with $u_0$ given by two characteristics. Computed with the numerical scheme for $m$ in Section 5 reproducing the exact solution up to approximation error. The function $u = m_\rho$ is recovered by numerical differentiation.

**Approximation by viscosity solutions**  We will prove there is an explicit solution defined for some $T > 0$ which is of viscosity type for $t > 0$. We will approximate the initial data by

$$
u_0^{\varepsilon,\delta} = \begin{cases} \dfrac{m_1}{\varepsilon} & \rho \in [\rho_1, \rho_1 + \varepsilon] \\[2mm] \dfrac{m_2}{\varepsilon}\dfrac{(\rho - (\rho_2 - \delta))}{\delta} & \rho \in [\rho_2 - \delta, \rho_2] \\[2mm] \dfrac{m_2}{\varepsilon} & \rho \in [\rho_2, \rho_2 + \varepsilon] \\[2mm] 0 & \text{otherwise} \end{cases}, \tag{4.24}
$$

for $\varepsilon$ and $\delta$ small enough. The $\varepsilon$-regularisation is used to approximate the Dirac deltas at the level of $u$. The $\delta$-regularisation is used to resolve the appearance of a rarefaction wave at $\rho_2$ due to a gap in the characteristics. Since viscosity solutions are stable by passage to the limit, we only need to show that our approximating solution are viscosity solutions.

The first part of the solutions does not notice the $\delta$-regularisation. We take $\varepsilon$ small enough so that $\rho_1 + \varepsilon < \rho_2 - 2\varepsilon$. For $\rho < \rho_2$ we reconstruct a vortex type solution following Section 4.2 with an initial gap

$$
u_\varepsilon(t, \rho) = \begin{cases} 0 & \rho < \rho_1 \\[2mm] \left( \left( \dfrac{m_1}{\varepsilon} \right)^{-\alpha} + \alpha t \right)^{-\frac{1}{\alpha}} & \rho \in [\rho_1, S_1^\varepsilon(t)] \end{cases} \tag{4.25}
$$

where the first shock is given by

$$
S_1^\varepsilon(t) = \rho_1 + \varepsilon_1 + m_1 \left( \left( \left( \dfrac{m_1}{\varepsilon} \right)^{-\alpha} + \alpha t \right)^{\frac{\alpha-1}{\alpha}} - \left( \dfrac{m_1}{\varepsilon} \right)^{1-\alpha} \right). \tag{4.26}
$$

Solutions in this form are defined for $t \in [0, T_\varepsilon)$ such that $S_1^\varepsilon(T_\varepsilon) = \rho_2 - \delta$. We leave to the reader to check that $T_\varepsilon$ does not tend to zero with $\varepsilon \to 0$.

For the second part, the characteristics with foot $\rho_0 \in [\rho_2 - \delta, \rho_2]$ are given by

$$
\rho = \rho_0 + \alpha \left( m_1 + \dfrac{m_2}{2} \dfrac{(\rho_0 + \delta - \rho_2)^2}{\varepsilon\delta} \right) \left( \dfrac{m_2}{\varepsilon\delta}(\rho_2 + \delta - \rho) \right)^{\alpha-1} t \tag{4.27}
$$

On the other hand, if $\rho_0 \in [\rho_2, \rho_2 + \varepsilon]$ we have

$$
\rho = \rho_0 + \alpha \left( m_1 + \dfrac{m_2}{2\varepsilon}\delta + \dfrac{m_2}{\varepsilon}(1 - (\rho_2 + \varepsilon - \rho_0)) \right) \left( \dfrac{m_2}{\varepsilon} \right)^{\alpha-1} t. \tag{4.28}
$$

Notice that in both cases $u$ is given by

$$
u(t, \rho) = (u_0(\rho_0)^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}.
$$

By mass conservation we have a further shock starting from $\rho_2 + \varepsilon$ given by a Rankine-Hugoniot condition

$$
\dfrac{dS_2^{\varepsilon,\delta}}{dt}(t) = (m_1 + m_2)u_{\varepsilon,\delta}(t, S_2^\varepsilon(t)^-)^{\alpha-1}.
$$

The first part of solution is of viscosity type, by an argument analogous to Section 4.2 and the second part have a monotone non-decreasing datum with final cut-off as in Section 4.3. We are reduced now to pass to the limit as $\varepsilon$ and $\delta$ tend to $0$.


**Passage to the limit as $\delta \to 0$**  For $[0, \rho_2 - \delta]$ the solution did not depend on $\delta$, so there is no work needed. Applying a similar argument as in [9] we can pass to the limit in (4.27). The characteristics with foot in $[\rho_2, \rho_2 + \delta]$ collapse to a rarefaction fan at $\rho_2$ of the form

$$
\rho = \rho_2 + m_1\eta_0^{\alpha-1}t, \qquad \eta_0 \in \left[ 0, \dfrac{m_2}{2\varepsilon} \right]. \tag{4.29}
$$

By inverting $\eta_0$ in (4.29) we recover the solution

$$u_\varepsilon(t, \rho) = \left(\eta_0^{-\alpha} + \alpha t\right)^{-\frac{1}{\alpha}} = \left(\left(\frac{\rho - \rho_2}{\alpha m_1 t}\right)^{-\frac{\alpha}{\alpha-1}} + \alpha t\right)^{-\frac{1}{\alpha}}.$$

The other characteristics are for foot $\rho_0 \in [\rho_2, \rho_2 + \varepsilon]$ and, by passing analogously to the limit in (4.28), we have

$$\rho = \rho_0 + \alpha \left(m_1 + \frac{m_2}{\varepsilon}(1 - (\rho_2 + \varepsilon - \rho_0))\right)\left(\frac{m_2}{\varepsilon}\right)^{\alpha-1} t.$$

Since $u_0^\varepsilon$ is non-decreasing the characteristics do not cross. The Rankine-Hugoniot condition is now

$$\frac{dS_2^\varepsilon}{dt}(t) = (m_1 + m_2)u_\varepsilon(t, S_2^\varepsilon(t)^-)^{\alpha-1}.$$

**Passage to the limit as** $\varepsilon \to 0$    Passing to the limit we end up only with the rarefaction fan characteristics and recover (4.19) where the first shock is given by (4.20) and the second shock, $S_2$ which defines the support, is a solution of the ODE

$$\begin{cases} \dfrac{dS_2}{dt}(t) = (m_1 + m_2)\left(\left(\dfrac{S_2(t) - \rho_2}{\alpha m_1 t}\right)^{-\frac{\alpha}{\alpha-1}} + \alpha t\right)^{-\frac{\alpha-1}{\alpha}}, \\ S_2(0) = \rho_2. \end{cases} \tag{4.30}$$

Notice that this equation is singular at $t = 0$ but it can be rewritten as

$$\frac{dS_2}{dt}(t) = (m_1 + m_2)t^{-\frac{\alpha-1}{\alpha}}\left(t^{\frac{1}{\alpha-1}}\left(\frac{S_2(t) - \rho_2}{\alpha m_1}\right)^{-\frac{\alpha}{\alpha-1}} + \alpha\right)^{-\frac{\alpha-1}{\alpha}}.$$

Since $\frac{\alpha-1}{\alpha} \in (0, 1)$ the Cauchy problem is well-posed. Alternatively, one can write $S_2$ implicitly as the only value such that

$$\int_{\rho_2}^{S_2(t)} u(t, \rho)\,\mathrm{d}\rho = m_2.$$

In other words,

$$\int_{\rho_2}^{S_2(t)}\left(\left(\frac{\rho - \rho_2}{\alpha m_1 t}\right)^{-\frac{\alpha}{\alpha-1}} + \alpha t\right)^{-\frac{1}{\alpha}}\,\mathrm{d}\rho = m_2. \tag{4.31}$$

This solution is defined for $0 \le t < T$ where

$$T = \frac{1}{\alpha}\left(\frac{\rho_2 - \rho_1}{m_1}\right)^{\frac{\alpha}{\alpha-1}}.$$

By scaling analysis on the integral, we can give an algebraic expression of $S_2(t)$. We apply the change of variables $\rho = \rho_2 + \alpha m_1 s t^{\frac{1}{\alpha}}$ to deduce

$$m_2 = \int_{\rho_2}^{S_2(t)}\left(\left(\frac{\rho - \rho_2}{\alpha m_1 t}\right)^{-\frac{\alpha}{\alpha-1}} + \alpha t\right)^{-\frac{1}{\alpha}}\,\mathrm{d}\rho = \int_0^{\frac{S_2(t)-\rho_2}{\alpha m_1 t^{\frac{1}{\alpha}}}}\left(\left(st^{-\frac{\alpha-1}{\alpha}}\right)^{-\frac{\alpha}{\alpha-1}} + \alpha t\right)^{-\frac{1}{\alpha}}t^{\frac{1}{\alpha}}\alpha m_1\,\mathrm{d}s$$

$$= \alpha m_1 \int_0^{\frac{S_2(t)-\rho_2}{\alpha m_1 t^{\frac{1}{\alpha}}}}\left(s^{-\frac{\alpha}{\alpha-1}} + \alpha\right)^{-\frac{1}{\alpha}}\,\mathrm{d}s = \alpha m_1 \mathcal{K}\left(\frac{S_2(t) - \rho_2}{\alpha m_1 t^{\frac{1}{\alpha}}}\right).$$

Hence, we recover (4.21). To show (4.23) we simply indicate that, for $s \le s_0$

$$s^{-\frac{\alpha}{\alpha-1}} \le s^{-\frac{\alpha}{\alpha-1}} + \alpha \le C(s_0)s^{-\frac{\alpha}{\alpha-1}}$$

## 4.5 Waiting time

### 4.5.1 Existence and estimate

We turn the explicit solution in (3.7) into a viscosity subsolution by extending it by zero, that is we define $\underline{m}(t, \rho)$ as

$$
\underline{m}(t,\rho) := \begin{cases} 0 & \text{if } \rho < c_0 - \alpha^{\frac{1}{\alpha}} M (T-t)^{\frac{1}{\alpha}} \\ \left( M^{\frac{\alpha}{\alpha-1}} - \alpha^{\frac{1}{\alpha-1}} \dfrac{(c_0-\rho)^{\frac{\alpha}{\alpha-1}}}{(T-t)^{\frac{1}{\alpha-1}}} \right)^{\frac{\alpha-1}{\alpha}} & \text{if } \rho \in (c_0 - \alpha^{\frac{1}{\alpha}} M(T-t)^{\frac{1}{\alpha}}, c_0) \\ M & \text{if } \rho \geq c_0 \end{cases} \tag{4.32}
$$

**Proposition 4.8.** $\underline{m}(t, \rho)$ *is a viscosity subsolution of* $m_t + m m_\rho^\alpha = 0$.

*Proof.* It is clear that $0$ is a solution of $m_t + m m_\rho^\alpha = 0$. So is the second part for $\rho > c_0 - c^{\frac{1}{\alpha}} M(T-t)^{\frac{1}{\alpha}}$, as we have checked in Section 3.4. At the matching point $\rho = c_0 - c^{\frac{1}{\alpha}} M(T-t)^{\frac{1}{\alpha}}$, we have that

$$
m_\rho(t, \rho^-) = 0, \qquad m_\rho(t, \rho^+) = c \left(0^+\right)^{-\frac{1}{\alpha}} \frac{(c_0-\rho)^{\frac{1}{\alpha}}}{(T-t)^{\frac{1}{\alpha}}} = +\infty
$$

This corner does not allow any smooth $\varphi$ to be tangent from above at this point, and hence the condition of viscosity subsolution is trivially satisfied. $\qquad\square$

We will denote by $c_0 = \max \operatorname{supp} u_0$, where $u_0 = (m_0)_\rho$, that coincides with the boundary of $m_0 = M$ in the sense that

$$
m_0(\rho) < M \text{ for } \rho < c_0 \quad \text{and} \quad m_0(\rho) = M \text{ for } \rho \geq c_0. \tag{4.33}
$$

**Corollary 4.9.** *Let* $m_0 \in \operatorname{BUC}([0, +\infty))$ *and let* $c_0 = \max \operatorname{supp} u_0$. *If*

$$
\limsup_{\rho \to c_0^-} \frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}} < +\infty, \tag{4.34}
$$

*then there is waiting time as in Corollary 3.5.*

*Proof.* First, we prove that

$$
\sup_{\rho \in [0, c_0]} \frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}} < +\infty.
$$

Let $\rho_k$ be such that

$$
\frac{M - m_0(\rho_k)}{(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}} \longrightarrow \sup_{\rho \in [0, c_0]} \frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}}.
$$

If the supremum were infinite, since $M - m_0(\rho_k)$ is bounded, then we have that $\rho_k \to c_0$. This results in

$$
\lim_k \frac{M - m_0(\rho_k)}{(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}} \leq \limsup_{\rho \to c_0^-} \frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}} < +\infty
$$

leading to a contradiction.

Therefore, there exists $C > 0$ such that for all $\rho \in [0, c_0]$

$$
\frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}} \leq C.
$$

In particular, we have that

$$
m_0(\rho) \geq M - C(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}.
$$

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We can apply the convexity of the function $f(x) = x^{\frac{\alpha}{\alpha-1}}$ to show that

$$m_0(\rho)^{\frac{\alpha-1}{\alpha}} \geq M^{\frac{\alpha}{\alpha-1}} - \frac{\alpha}{\alpha-1} M^{\frac{1}{\alpha-1}} C(c_0 - \rho)^{\frac{\alpha}{\alpha-1}} = M^{\frac{\alpha}{\alpha-1}} - \frac{\alpha^{\frac{1}{\alpha-1}}}{T^{\frac{1}{\alpha-1}}}(c_0 - \rho)_+^{\frac{\alpha}{\alpha-1}} = \underline{m}(0, \rho)^{\frac{\alpha-1}{\alpha}}.$$

for a well chosen $T$ (see, e.g. Figure 6). Therefore, applying the comparison principle Theorem 4.3 then $m \geq \underline{m}$, and thus $m$ has waiting time. □
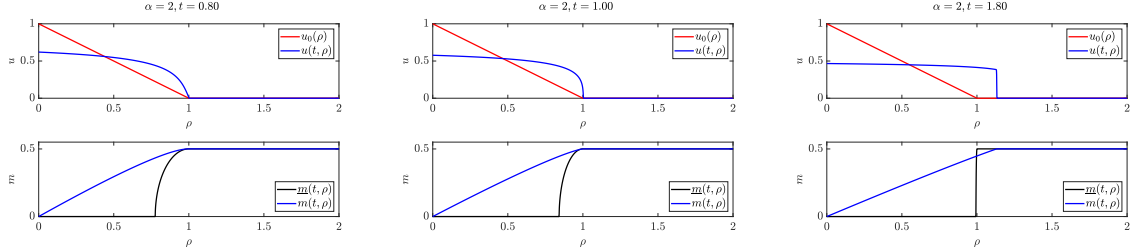


Figure 6: The explicit Ansatz viscosity subsolution (4.32) guarantees existence of waiting time. The subsolution is represented from the explicit solution, whereas $u$ is computed through the numerical scheme in Section 5. See a movie simulation in the supplementary material [20, Video 1].

### 4.5.2 Non-existence of waiting time

**Theorem 4.10.** *Let $m_0 \in \mathrm{BUC}([0, +\infty))$ and let $c_0 = \max \mathrm{supp}\, u_0$. If*

$$\limsup_{\rho \to c_0^-} \frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}} = +\infty. \tag{4.35}$$

*Then, there is no waiting time.*

*Proof.* To prove there is no waiting time, we want to show that $S(t) > c_0$ for any $t > 0$. In order to do this, we will construct we a sequence of supersolutions $\overline{m}_k$ with $S_k(t) = \max \mathrm{supp}(\overline{m}_k)_\rho(t, \cdot)$ and times $t_k \searrow 0$ such that $S_k(t_k) > c_0$. This ensures that, for some $k$ we have $0 < t_k < t$ and hence $S(t) \geq S_k(t) \geq S_k(t_k) > c_0$.

Let us consider a sequence $d_k$ such that

$$d_k \to \limsup_{\rho \to c_0^-} \frac{M - m_0(\rho)}{(c_0 - \rho)^{\frac{\alpha}{\alpha-1}}}.$$

There exists $\rho_k \nearrow c_0$ such that

$$M - m_0(\rho_k) \geq d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}. \tag{4.36}$$

We construct the viscosity supersolutions $\overline{m}_k$ with initial derivative

$$u_{k,0} = m_0(\rho_k)\delta_0 + (M - m_0(\rho_k))\delta_{\rho_k}.$$

It is clear that $\overline{m}_k(0, \rho) \geq m(0, \rho)$. By using the comparison principle Theorem 4.3, $\overline{m}_k \geq m$.

Now we apply the theory of Section 4.4. We will select $t_k > 0$ such that $S_k(t) \geq c_0 + \varepsilon_k$ for $t \geq t_k$ with $\varepsilon_k = \frac{c_0 - \rho_k}{2} > 0$. Using (4.21), (4.22) and (4.36) we have that

$$S_k(t) = \rho_k + \alpha m_0(\rho_k)\mathcal{K}^{-1}\left(\frac{M - m_0(\rho_k)}{\alpha m_0(\rho_k)}\right) t^{\frac{1}{\alpha}} \geq \rho_k + \alpha m_0(\rho_k)\mathcal{K}^{-1}\left(\frac{d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}}{\alpha m_0(\rho_k)}\right) t^{\frac{1}{\alpha}}.$$

Due to our choice of $\rho_k$, it is sufficient that

$$\rho_k + \alpha m_0(\rho_k)\mathcal{K}^{-1}\left(\frac{d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}}{\alpha m_0(\rho_k)}\right) t^{\frac{1}{\alpha}} \geq c_0 + \varepsilon_k.$$

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Solving for $t$, we have that

$$t \geq \left( \frac{\frac{(c_0 - \rho_k) + \varepsilon_k}{\alpha m_0(\rho_k)}}{\mathcal{K}^{-1}\left(\frac{d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}}{\alpha m_0(\rho_k)}\right)} \right)^{\alpha}. \tag{4.37}$$

We know that $d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}} \leq M - m_0(\rho_k) \to 0$, therefore we need to study $\mathcal{K}^{-1}$ close to $0$. Going back to (4.23) there exists a constant $C > 0$ such that

$$\left( \frac{\frac{(c_0 - \rho_k) + \varepsilon_k}{\alpha m_0(\rho_k)}}{\mathcal{K}^{-1}\left(\frac{d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}}{\alpha m_0(\rho_k)}\right)} \right)^{\alpha} \geq C \left( \frac{\frac{(c_0 - \rho_k) + \varepsilon_k}{\alpha m_0(\rho_k)}}{\left(\frac{d_k(c_0 - \rho_k)^{\frac{\alpha}{\alpha-1}}}{\alpha m_0(\rho_k)}\right)^{\frac{\alpha-1}{\alpha}}} \right)^{\alpha} = \frac{3^{\alpha}}{2^{\alpha}} \frac{C}{d_k^{\alpha-1}} =: t_k.$$

Due to the hypothesis of the theorem $d_k \to +\infty$ and hence $t_k \to 0$. $\qquad\square$

See a movie simulation of one of the mass supersolutions interacting with a solution without waiting time in the supplementary material [20, Video 2].

**Remark 4.11.** Notice that if the $\limsup$ is finite, the previous proof can be adapted to show that the supersolutions $\overline{m}_k$ give an upper bound of the waiting time.

**Remark 4.12.** As pointed out in Corollary 3.5, the spatial support of classical solutions does not change in time. Taking $c_0 = \max \operatorname{supp} u_0$, we construct the supersolution $\overline{m}$ with initial derivative

$$\overline{u}_0 = m_0\left(\frac{c_0}{2}\right)\delta_0 + \left(M - m_0\left(\frac{c_0}{2}\right)\right)\delta_{\frac{c_0}{2}}.$$

This supersolution shows that the support of $u$ must move after a finite time and therefore that the solution is no longer classical.

## 4.6 Asymptotic behaviour

We give first a general result of asymptotic behaviour in mass variable.

**Theorem 4.13.** *Assume that $u_0 \in L^{\infty}(0, \infty)$ has compact support, $M = \|u_0\|_{L^1}$, $m$ be the viscosity solution with initial data $m_0$ and $S(t) = \inf\{\rho : m(t, \rho) = M\}$. Then $S(t) \sim M(\alpha t)^{\frac{1}{\alpha}}$ with estimate*

$$0 \leq \frac{S(t)}{M(\alpha t)^{\frac{1}{\alpha}}} - 1 \leq \frac{S(0)}{M}(\alpha t)^{-\frac{1}{\alpha}}. \tag{4.38}$$

*Furthermore, $m$ has the asymptotic profile in rescaled variable $y = \frac{\rho}{M(\alpha t)^{\frac{1}{\alpha}}}$ with an asymptotic estimate*

$$\sup_{y \geq \varepsilon} \left| \frac{m\left(t, M(\alpha t)^{\frac{1}{\alpha}} y\right)}{MG(y)} - 1 \right| \to 0, \quad \text{as } t \to +\infty \quad \text{where } G(y) = \begin{cases} y & y \leq 1 \\ 1 & y > 1 \end{cases} \tag{4.39}$$

*for any $\varepsilon > 0$.*

*Proof.* By Remark 4.6 we take as super and subsolution $\overline{m}$ and $\underline{m}$ with initial data

$$\overline{m}_0(\rho) = MH_0(\rho), \qquad \underline{m}_0(\rho) = MH_{S(0)}(\rho).$$

Hence $\overline{m} \geq m \geq \underline{m}$. Due to the explicit form of $\overline{m}$ and $\underline{m}$, we have that

$$M(\alpha t)^{\frac{1}{\alpha}} \leq S(t) \leq S(0) + M(\alpha t)^{\frac{1}{\alpha}}.$$

Due to the self-similar form of $\overline{m}$ and $\underline{m}$ given in Remark 4.6, the result is proven. $\qquad\square$

**Remark 4.14.** Notice that for $m_0 = 0$ in $[0, a]$, we have $m(t, \rho) = 0$ in $[0, a]$ so the supremum of $y \geq 0$ is always 1. If $u_0$ is continuous and $u_0(0) > 0$, then the supremum can be taken for $y \geq 0$.

Let us discuss the asymptotic behaviour when the datum is monotone non-decreasing with final cut-off. We recall (4.14)-(4.17). We define

$$U(t, \xi) = \frac{u(t, (\alpha t)^{\frac{1}{\alpha}} \xi)}{(\|u_0\|_{L^\infty}^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}}$$

Since the solution is constructed by characteristics we have that

$$U(t, \xi) = \begin{cases} 0 & \text{if } u_0((\alpha t)^{-\frac{1}{\alpha}} \xi) = 0 \\ \left( \dfrac{\eta_0(t, \xi)^{-\alpha} + \alpha t}{\|u_0\|_{L^\infty}^{-\alpha} + \alpha t} \right)^{-\frac{1}{\alpha}} & \text{if } u_0((\alpha t)^{-\frac{1}{\alpha}} \xi) > 0 \text{ and } (\alpha t)^{\frac{1}{\alpha}} \xi \leq S(t) \\ 0 & \text{if } (\alpha t)^{\frac{1}{\alpha}} \xi > S(t) \end{cases}$$

where $\eta_0(t, \xi) \in (0, \|u_0\|_{L^\infty})$. Due to (4.38), as $t \to +\infty$ we have that

$$U(t, \xi) \to \begin{cases} 1 & \text{if } \xi \in (0, M) \\ 0 & \text{if } \xi \in (M, +\infty). \end{cases}$$

The value at $0$ depends on whether $u_0(0) = 0$.

# 5  A numerical scheme

In the pioneering paper by Crandall and Lions [10], the authors developed a theory of monotone schemes for finite differences of Hamilton-Jacobi equations, where solutions are shown to converge to the viscosity solution. They study equations of the form

$$m_t + H(m_\rho) = 0. \tag{5.1}$$

For these equations it is natural to develop only explicit methods. However, for our case $m_t + H(m_\rho)m = 0$, we will see that it more natural, and probably more stable, to do an explicit-implicit approximation of the non-linear term $H(m_\rho)m$. In fact, since the nonlinear term is linear in $m$, we can solve for the implicit step in an explicit manner. More precisely, considering an equispaced discretization

$$t_n = h_t n \qquad \rho_j = h_\rho j. \tag{5.2}$$

We select the following finite-difference schemes

$$\frac{M_j^{n+1} - M_j^n}{h_t} + \left( \frac{M_j^n - M_{j-1}^n}{h_\rho} \right)^\alpha M_j^{n+1} = 0$$

which can be written as

$$M_j^{n+1} = \frac{M_j^n}{1 + h_t \left( \dfrac{M_j^n - M_{j-1}^n}{h_\rho} \right)_+^\alpha} = G(M_j^n, M_{j-1}^n). \tag{5.3}$$

Here, $G$ is given by

$$G(p, q) = \frac{p}{1 + h_t H \left( \frac{p-q}{h_\rho} \right)}, \qquad \text{where } H(s) = s_+^\alpha.$$

Notice that the method depends only on the parameter $h_t / h_\rho^\alpha$. Taking derivatives we have that

$$\frac{\partial G}{\partial p} = \frac{1 + h_t H \left( \frac{p-q}{h_\rho} \right) - \frac{h_t}{h_\rho} H' \left( \frac{p-q}{h_\rho} \right) p}{\left( 1 + h_t H \left( \frac{p-q}{h_\rho} \right) \right)^2}, \qquad \frac{\partial G}{\partial q} = \frac{p h_t H' \left( \frac{p-q}{h_\rho} \right)}{h_\rho \left( 1 + h_t H \left( \frac{p-q}{h_\rho} \right) \right)^2} \geq 0$$

Then $G$ is non-decreasing in $p$ under the simple CFL condition:

$$\frac{h_t}{h_\rho} H'\left(\frac{p-q}{h_\rho}\right) p \le \frac{1}{2}. \tag{5.4}$$

Since the denominator in $G$ is larger than $1$, we have that $G(p,q) \le p$. This is immediately translated to a maximum principle for $M_j^n$

$$M_j^{n+1} \le M_j^n \le \|m_0\|_\infty. \tag{5.5}$$

For $\alpha \ge 1$ we have two options to obtain a CFL condition. We can check whether the numerical derivative is bounded (this can be done for some methods, see Section 5.2) or cut-off the equation by a nice value. For $m_0$ fixed, since $m_\rho \le \|(m_0)_\rho\|_{L^\infty}$ due to (4.4), the equation (5.1) where $H(s) = s_+^\alpha$ is equivalently to itself with

$$H(s) = (\max\{s, \|(m_0)_\rho\|_\infty\})_+^\alpha. \tag{5.6}$$

We write this cut-off to ensure monotonicity. Nevertheless, once the method is monotone, Lemma 5.3 ensures that the cut-off part is not reached. Hence, this cut-off is purely technical.

For $\alpha \ge 1$ this new $H$ given by (5.6) satisfies

$$0 \le H'(s) \le \alpha\|(m_0)_\rho\|_\infty^{\alpha-1}.$$

Therefore, (5.4) can be taken as

$$\frac{h_t}{h_\rho} \le \frac{1}{2\alpha\|(m_0)_\rho\|_\infty^{\alpha-1}\|m_0\|_\infty}. \tag{CFL}$$

We propose the scheme

$$\begin{cases} M_j^{n+1} = \dfrac{M_j^n}{1 + h_t H\left(\dfrac{M_j^n - M_{j-1}^n}{h_\rho}\right)} & \text{if } j > 0, n \ge 0 \\ M_j^0 = m_0(h_\rho j) & \text{if } j \ge 0 \\ M_0^n = 0 & \text{if } n > 0. \end{cases} \tag{M}$$

**Remark 5.1.** As we pointed out in [9], for $0 < \alpha < 1$ this method is not monotone. This was fixed by regularising $H$. For $\delta > 0$ we take

$$H_\delta(s) = (s_+ + \delta)^\alpha - \delta^\alpha. \tag{5.7}$$

By including the boundary and initial condition, we constructed the method

$$\begin{cases} M_j^{n+1} = \dfrac{M_j^n}{1 + h_t H_\delta\left(\dfrac{M_j^n - M_{j-1}^n}{h_\rho}\right)} & \text{if } j > 0, n \ge 0 \\ M_j^0 = m_0(h_\rho j) & \text{if } j \ge 0 \\ M_0^n = 0 & \text{if } n > 0. \end{cases} \tag{M$_\delta$}$$

with this regularisation we know that $H_\delta'(s) \le \alpha\delta^{\alpha-1}$ so we have a CFL condition

$$\frac{h_t}{h_\rho} \le \frac{\delta^{1-\alpha}}{2\alpha\|m_0\|_\infty}. \tag{CFL$_\delta$}$$

In [9] we made $\delta$ to converge to $0$ with $h_t$ and $h_\rho$, showing the convergence of the numerical solutions.

## 5.1 Properties of monotone methods

The following properties of (M) when $G$ is monotone in each variable are a classical matter (see the original result in [10] and the presentation and references in [1]). We just briefly sketch them for completeness.

**Lemma 5.2.** *Let $\alpha \geq 1$, $m_0 \geq 0$ and bounded and consider the sequence $M_j^n$ constructed by (M) and assume (CFL). We have the following properties:*

1. $M_j^{n+1} = G(M_j^n, M_{j-1}^n)$ *where $G$ is non-decreasing.*

2. $M_j^n \leq \|m_0\|_\infty$

3. *If $m_0 \geq 0$ is non-decreasing then:*

    (a) $0 \leq M_j^n \leq M_{j+1}^n$ *for all $n, j$*

    (b) *There is mass conservation in the numerical scheme*

    $$M_\infty^{n+1} = \lim_{j \to +\infty} M_j^{n+1} = \lim_{j \to +\infty} M_j^n = M_\infty^n.$$

*Proof.*     1. We have shown this above.

2. This is true for $M_j^0$ by construction, and hence for all $n$, due to the previous item.

3.    (a) We proceed by induction in $n$. For time $n = 0$ this is true due to the monotonicity of $m_0$. Assume $M_j^n \leq M_{j+1}^n$ for all $j$. Since $G$ is monotone in each coordinate

    $$M_{j+1}^{n+1} = G(M_{j+1}^n, M_j^n) \geq G(M_j^n, M_j^n) \geq G(M_j^n, M_{j-1}^n) = M_j^{n+1}.$$

    (b) Since the sequence is non-decreasing and bounded, it has a limit. Furthermore $\lim_j(M_j^n - M_{j-1}^n) = 0$. Hence, since $H_\delta(0) = 0$ we have that

    $$M_\infty^{n+1} = \lim_{j \to +\infty} M_j^{n+1} = \lim_{j \to +\infty} \frac{M_j^n}{1 + h_t H\left(\dfrac{M_j^n - M_{j-1}^n}{h_\rho}\right)} = M_\infty^n. \qquad \square$$

Notice the biggest advantage of the method (M) is that it preserves the space monotonicity of $m$ and the total mass, as it should be for a mass equation.

## 5.2 Convergence of the numerical scheme (M) to the viscosity solution

In order to construct a convergent scheme, it is better to work with a single parameter. For $h > 0$ we define

$$h_\rho = h, \qquad h_t = \frac{h}{2\alpha \|(m_0)_\rho\|_\infty^{\alpha-1} \|m_0\|_\infty}.$$

so that (CFL) is satisfied. We now allow $M_j^n$ to be constructed from (M). For $t_n \leq t < t_{n+1}$ and $\rho_j \leq \rho < \rho_{j+1}$ we write the piecewise linear interpolation of the discrete values as

$$m^h(t, \rho) = \begin{cases} M_j^n + (\rho - \rho_j)\dfrac{M_{j+1}^n - M_j^n}{h_\rho} + (t - t_n)\dfrac{M_j^{n+1} - M_j^n}{h_t} & \text{if } \rho \leq t \\[4mm] M_{j+1}^{n+1} - (\rho_{j+1} - \rho)\dfrac{M_{j+1}^{n+1} - M_j^{n+1}}{h_\rho} - (t_{n+1} - t)\dfrac{M_{j+1}^{n+1} - M_{j+1}^n}{h_t} & \text{if } \rho > t \end{cases} \tag{5.8}$$

This construction ensures that

$$\frac{\partial m^h}{\partial \rho} = \begin{cases} U_{j+1}^n & \text{if } \rho < t \\ U_{j+1}^{n+1} & \text{if } \rho > t \end{cases}, \qquad \frac{\partial m^h}{\partial t} = \begin{cases} -H\left(U_j^n\right) M_j^{n+1} & \text{if } \rho < t \\ -H\left(U_{j+1}^n\right) M_{j+1}^{n+1} & \text{if } \rho > t \end{cases}$$

where $U_j^n$ is the numerical space derivative

$$U_j^n = \frac{M_j^n - M_{j-1}^n}{h_\rho} \geq 0, \tag{5.9}$$

and the numerical time derivative is given by the relation

$$\frac{M_j^{n+1} - M_j^n}{h_t} = -H\left(U_j^n\right) M_j^{n+1} \leq 0.$$

The strategy of the proof is the following. We will show that these space and time numerical derivatives are uniformly bounded, and hence $m^h$ is uniformly continuous, non-decreasing in $\rho$ and non-increasing in $t$. We can then apply the Ascoli-Arzelá precompactness theorem and show there exists a convergent subsequence. We will prove the limit is the viscosity solution.

If we subtract (M) for $j$ and $j-1$ we recover an equation for the numerical derivative $U_j^n$

$$\frac{U_j^{n+1} - U_j^n}{h_t} + \frac{H(U_j^n)M_j^{n+1} - H(U_{j-1}^n)M_{j-1}^{n+1}}{h_\rho} = 0. \tag{5.10}$$

Notice that the natural scaling for this equation is $h_t/h_\rho$.

### 5.2.1   Boundedness of the numerical derivative

Since (5.10) is a numerical approximation by a monotone method of the nonlinear conservation law (P), we can expect a maximum principle.

**Lemma 5.3.** *Let* $0 \leq m_0$ *be uniformly Lipschitz continuous, bounded and non-decreasing,* $M_j^n$ *be given by* (M)*, that* (CFL) *holds and let* $U_j^n$ *given by* (5.9)*. Then,* $U_j^n \geq 0$ *and*

$$\sup_j U_j^{n+1} \leq \sup_j U_j^n \qquad \forall n \geq 0. \tag{5.11}$$

**Remark 5.4.** Once this is proven, the cut-off (5.6) is not needed.

*Proof.* That $U_j^n \geq 0$ follows form Lemma 5.2. We write

$$
\begin{aligned}
0 &= \frac{U_j^{n+1} - U_j^n}{h_t} + M_{j-1}^{n+1}\frac{H(U_j^n) - H(U_{j-1}^n)}{h_\rho} + H(U_j^n)\frac{M_j^n - M_{j-1}^n}{h_\rho} \\
&= \frac{U_j^{n+1} - U_j^n}{h_t} + M_{j-1}^{n+1}\frac{H(U_j^n) - H(U_{j-1}^n)}{h_\rho} + H(U_j^n)U_j^n.
\end{aligned}
$$

Solving for $U_j^{n+1}$, using the fact that $H$ is non-decreasing and $U_j^n \geq 0$, we have that

$$
\begin{aligned}
U_j^{n+1} &\leq U_j^n - \frac{h_t}{h_\rho}M_{j-1}^{n+1}(H(U_j^n) - H(U_{j-1}^n)) \\
&= U_j^n - \frac{h_t}{h_\rho}M_j^{n+1}H'(\xi_j^n)U_j^n + \frac{h_t}{h_\rho}M_j^{n+1}H'(\xi_j^n)U_{j-1}^n \\
&= \left(1 - \frac{h_t}{h_\rho}M_j^{n+1}H'(\xi_j^n)\right)U_j^n + \frac{h_t}{h_\rho}M_j^{n+1}H'(\xi_j^n)U_{j-1}^n.
\end{aligned}
$$

Due to (CFL) we have that the coefficients in front of $U_j^n$ and $U_{j-1}^n$ are non-negative. Hence

$$
\begin{aligned}
U_j^{n+1} &\leq \left(1 - \frac{h_t}{h_\rho}M_j^{n+1}H'(\xi_j^n)\right)\sup_j U_j^n + \frac{h_t}{h_\rho}M_j^{n+1}H'(\xi_j^n)\sup_j U_j^n \\
&= \sup_j U_j^n.
\end{aligned}
$$

And this holds for every $j$ so the result is proved. $\qquad\square$

### 5.2.2  Convergence via Ascoli-Arzelá. Existence of a viscosity solution

**Theorem 5.5.** *Let us $m_0 \in W^{1,\infty}(0, +\infty)$ and non-decreasing,* (CFL)*, $M_j^n$ constructed by* (M) *and $m^h$ be given by* (5.8)*. Then, $m^h$ is a family of uniformly continuous functions. Then, for every $P > 0$*

$$m^h \to m \qquad \text{in } \mathcal{C}([0, P] \times [0, T]) \text{ as } h_\rho \to 0. \tag{5.12}$$

*where $m$ is a viscosity solution of* (4.2)*. Furthermore,* (4.4) *holds.*

*Proof.*  First, we notice that $m^h$ satisfies (4.4). Due to (5.8), we have that

$$|m_\rho^h(t, \rho)| \leq \|(m_0)_\rho\|_\infty, \qquad |m_t^h(t, \rho)| \leq |H(U_j^n)M_j^{n+1}| \leq H(\|(m_0)_\rho\|_\infty)\|m_0\|_\infty.$$

By the Ascoli-Arzelá theorem there is a subsequence that converges uniformly in $[0, P] \times [0, T]$.

We will show every convergent subsequence converges to the same limit $m$, and hence the whole sequence converges. We still denote by $h$ the indexes of the convergent subsequences.

Let $m^h$ be a subsequence converging in $[0, T] \times [0, P]$. We check that it is a viscosity subsolution, and the proof of viscosity supersolution is analogous. Let $(t_0, \rho_0) \in (0, T) \times (0, P)$ and $\varphi \in C^2$ be such that $m - \varphi$ has a strict local maximum at $(t_0, \rho_0)$ and $m(t_0, \rho_0) = \varphi(t_0, \rho_0)$. We can modify $\varphi$ outside a bounded neighbourhood of $(t_0, \rho_0)$, so that $m - \varphi$ attains a unique global maximum at $(t_0, \rho_0)$, for $h$ large enough $m^h - \varphi$ attains a global maximums in $[0, T] \times [0, P]$ at an interior points $(t_h, \rho_h)$, and $(t_h, \rho_h) \to (t_0, \rho_0)$ as $h \to 0$. Our argument is a variation of [18, Lemma 1.8].

Let $B \subset [0, T] \times [0, P]$ be a small open ball around $(t_0, \rho_0)$ where the maximum is global. Let $\varepsilon = -\inf_B(m - \varphi)/2$. Define $U = \{m - \varphi > -\varepsilon\} \cap B$ which is a open and bounded neighbourhood of $(t_0, \rho_0)$. We modify $\varphi$ so that is greater than $m + \varepsilon$ outside $U$. With the modification, $m - \varphi$ attains a unique global maximum at $(t_0, \rho_0)$.

Let $h$ be small enough so that $|m^h - m| < \frac{\varepsilon}{2}$ in $[0, T] \times [0, P]$. We have that

$$\max_{[0, t_0+1] \times [0, \rho_0+1] \setminus U} (m^h - \varphi) < \max_{[0, t_0+1] \times [0, \rho_0+1] \setminus U} (m - \varphi) + \frac{\varepsilon}{2} \leq -\frac{\varepsilon}{2}.$$

On the other hand

$$m^h(t_0, \rho_0) - \varphi(t_0, \rho_0) > m(t_0, \rho_0) - \varphi(t_0, \rho_0) - \frac{\varepsilon}{2} = -\frac{\varepsilon}{2}.$$

Therefore the maximum over $[0, T] \times [0, P]$ is attained at some $(t_h, \rho_h) \in U$. The sequence $(t_h, \rho_h)$ is bounded, and therefore as a convergent subsequence. Let $(t_1, h_1)$ be its limit. We have that

$$m^h(t_h, \rho_h) - \varphi(t_h, \rho_h) \geq m^h(t, \rho) - \varphi(t, \rho) \qquad \forall (t, \rho) \in [0, T] \times [0, P].$$

Passing to the limit, since the maximum is unique, we have that $(t_1, \rho_1) = (t_0, \rho_0)$. Since all convergent subsequences share a limit, the whole sequence converges.

For such small values of $h$, let us define

$$n_h = \left\lfloor \frac{t_h}{h_t} \right\rfloor - 1, \qquad j_h = \left\lceil \frac{\rho_h}{h_\rho} \right\rceil.$$

Since $m^h - \varphi$ has a global maximum in $[0, T] \times [0, P]$, we have that

$$m^h(t_h, \rho_h) - \varphi(t_h, \rho_h) \geq m^h(t, \rho) - \varphi(t, \rho).$$

Evaluating on the points of the mesh, we get that

$$M_j^n \leq \varphi(t_n, \rho_j) - \varphi(t_h, \rho_h) + m^h(t_h, \rho_h).$$

Since $m^h$ is increasing in $\rho$ and decreasing in $t$ and the fact that $G$ is non-decreasing, we recover

$$m^h(t_h, \rho_h) \leq m^h((n_h + 1)h_t, j_h h_\rho) = M_{j_h}^{n_h+1} = G(M_{j_h}^{n_h}, M_{j_h-1}^{n_h})$$

$$\leq G\Big(\varphi(t_{n_h}, \rho_{j_h}) - \varphi(t_h, \rho_h) + m^h(t_h, \rho_h), \varphi(t_{n_h}, \rho_{j_h-1}) - \varphi(t_h, \rho_h) + m^h(t_h, \rho_h)\Big)$$

$$= \frac{\varphi(t_{n_h}, \rho_{j_h}) - \varphi(t_h, \rho_h) + m^h(t_h, \rho_h)}{1 + h_t H\left(\frac{\varphi(t_{n_h}, \rho_{j_h}) - \varphi(t_{n_h}, \rho_{j_h-1})}{h_\rho}\right)}$$

due to the definition of $G$. Since $\varphi$ is smooth, for $h$ small enough the denominator is positive and hence

$$\frac{\varphi(t_h, \rho_h) - \varphi(t_{n_h}, \rho_{j_h})}{h_t} + H\left(\frac{\varphi(t_{n_h}, \rho_{j_h}) - \varphi(t_{n_h}, \rho_{j_h-1})}{h_\rho}\right) m^h(t_h, \rho_h) \le 0.$$

Adding and subtracting $\varphi(t_{n_h+1}, \rho_j)/h_\rho$ on both sides

$$\frac{\varphi(t_{n_h+1}, \rho_{j_h}) - \varphi(t_{n_h}, \rho_{j_h})}{h_t} + H\left(\frac{\varphi(t_{n_h}, \rho_{j_h}) - \varphi(t_{n_h}, \rho_{j_h-1})}{h_\rho}\right) m^h(t_h, \rho_h)$$

$$\le \frac{\varphi(t_{n_h+1}, \rho_{j_h}) - \varphi(t_h, \rho_h)}{t_h - t_{n_h}} \frac{t_h - t_{n_h}}{h_t}.$$

Clearly $t_h - t_{n_h} \ge 0$ and, since $\varphi$ is of class $C^1$ and $m$ is non-increasing in $t$, we have that

$$\lim_{h \to 0} \frac{\varphi(t_{n_h+1}, \rho_{j_h}) - \varphi(t_h, \rho_h)}{t_h - t_{n_h}} = \varphi_t(t_0, \rho_0) \le 0,$$

Therefore, as $h \to 0$, we conclude that

$$\varphi_t(t_0, \rho_0) + H\left(\varphi_\rho(t_0, \rho_0)\right) m(t_0, \rho_0) \le 0,$$

for any $\varphi$ such that $m - \varphi$ has a global maximum at $(t_0, \rho_0)$. Therefore, $m$ is a viscosity subsolution.
□

## 5.3 Rate of convergence

**Theorem 5.6.** Let $\alpha \ge 1$ and let $h_t$ and $h_\rho$ satisfy (CFL). Let $m_0$ be Lipschitz continuous and bounded and let $m$ be the viscosity solution of (4.2) and $M_j^n$ be constructed by (M). Then, for any $T > 0$

$$\sup_{\substack{j \ge 0 \\ 0 \le n \le T/h_t}} |m(t_n, \rho_j) - M_j^n| \le Ch_\rho^{\frac{1}{3}}.$$

where $C$ does not depend on $h_\rho$.

**Remark 5.7.** The original paper by Crandall and Lions allows, by a longer and more involved argument, proves estimates of the form $O(\sqrt{h_t})$ with $H$ continuous, but requiring that the function defining the method is Lipschitz continuous.

*Proof.* For convenience, in the proof we denote $N = \lceil T/h_t \rceil$. Our aim is to prove that

$$\sigma = \sup_{\substack{j \ge 0 \\ 0 \le n \le T/h_t}} (m(t_n, \rho_j) - M_j^n) \le Ch_\rho^{\frac{1}{3}}.$$

The argument can be analogously repeated for the infimum. If $\sigma \le 0$ there is nothing to prove. Assume that $\sigma > 0$. Let $L$ be the Lipschitz constant of $m_0$. Due to (4.4), it is also the Lipschitz constant of $m$.

We begin by indicating there exist $n_1, j_1$ such that

$$m(t_1, \rho_1) - M_{j_1}^{n_1} \ge \frac{3\sigma}{4}, \qquad \text{where } t_1 = h_t n_1 \text{ and } \rho = h_\rho j_1.$$

We define

$$\Phi(t, h_t n, \rho, h_\rho j) = m(t, \rho) - M_j^n - \phi(t, h_t n, \rho, h_\rho j)$$

where, for $\varepsilon, \lambda > 0$ we define

$$\phi(t, s, \rho, \xi) = \left(\frac{|\rho - \xi|^2 + |t - s|^2}{\varepsilon^2} + \lambda(t + s)\right)$$

Then the maximum is at $t_\varepsilon \in [0, T], \rho_\varepsilon \in [0, +\infty)$ and $t_\varepsilon = h_t n_\varepsilon$ with $n_\varepsilon \in \{0, \cdots, N\}, \xi_\varepsilon = h_\rho j_\varepsilon$ with $j_\varepsilon \in \mathbb{N} \cup \{0\}$. Again this function is continuous and

1. Defined over a bounded set in $t_\varepsilon$ and $s_\varepsilon$.

2. If $\rho \to +\infty$ and $j$ remains bounded then $\Phi \to -\infty$ (analogously in $\rho$ bounded and $j \to +\infty$).

3. If $\rho \to +\infty$ and $j \to +\infty$ then
$$\limsup_{\rho, j \to +\infty} \Phi \leq m_\infty - m_\infty \leq 0,$$

so there exists a point of maximum $(t_\varepsilon, h_t n_\varepsilon, \rho_\varepsilon, h_\rho j_\varepsilon)$ such that

$$\Phi(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) \geq \Phi(t, s, \rho, \xi) \qquad \forall (t, s, \rho, \xi).$$

In particular
$$\Phi(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) \geq \Phi(t_1, t_1, \rho_1, \rho_1) = m(t_1, \rho_1) - M_{j_1}^{n_1} - 2\lambda t_1.$$

Taking
$$\lambda = \frac{\sigma}{8(1+T)} \tag{5.13}$$

we have
$$\Phi(t_\varepsilon, \rho_\varepsilon, n_\varepsilon, j_\varepsilon) \geq \frac{\sigma}{2}.$$

In particular,
$$m(t_\varepsilon, \rho_\varepsilon) - M_{j_\varepsilon}^{n_\varepsilon} \geq \frac{\sigma}{2} + \phi(t_\varepsilon, h_t n_\varepsilon, \rho_\varepsilon, h_\rho j_\varepsilon) > 0 \tag{5.14}$$

**Step 1. Variables collapse.** As $\Phi(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) \geq \Phi(0,0,0,0) = 0$, we have

$$\frac{|\rho_\varepsilon - \xi_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2}{\varepsilon^2} + \lambda(t_\varepsilon + s_\varepsilon) \leq m(t_\varepsilon, \rho_\varepsilon) - M_{j_\varepsilon}^{n_\varepsilon} \leq 2\|m_0\|_\infty.$$

Therefore, we obtain

$$|\rho_\varepsilon - \xi_\varepsilon|^2 + |t_\varepsilon - s_\varepsilon|^2 \leq 2\|m_0\|_\infty \varepsilon^2, \qquad \text{and} \qquad \rho_\varepsilon^2 + \xi_\varepsilon^2 \leq \frac{2\|m_0\|_\infty}{\varepsilon}.$$

This implies that, as $\varepsilon \to 0$, the variable doubling collapses to a single point.

**Step 2. For $\varepsilon$ small enough, $t_\varepsilon, \rho_\varepsilon, n_\varepsilon, j_\varepsilon > 0$.** Since $m$ is Lipschitz continuous

$$\begin{aligned}
\frac{\sigma}{2} &< m(t_\varepsilon, \rho_\varepsilon) - M_{j_\varepsilon}^{n_\varepsilon} \\
&= m(t_\varepsilon, \rho_\varepsilon) - m(0, \rho_\varepsilon) + m(0, \rho_\varepsilon) - m(0, \xi_\varepsilon) \\
&\quad + m(0, \xi_\varepsilon) - M_{j_\varepsilon}^0 + M_{j_\varepsilon}^0 - M_{j_\varepsilon}^{n_\varepsilon} \\
&\leq Lt_\varepsilon + L|\rho_\varepsilon - \xi_\varepsilon|,
\end{aligned}$$

using the fact that $m(0, \xi_\varepsilon) = M_{j_\varepsilon}^0$ and $M_j^n$ is decreasing in $n$. If $\varepsilon$ is small enough

$$\varepsilon < \frac{\sigma}{4L\sqrt{2\|m_0\|_\infty}}, \tag{5.15}$$

we have $Lt_\varepsilon > \sigma/4$ and hence $t_\varepsilon > 0$. Analogously for $\rho_\varepsilon > 0$.

If $n_\varepsilon = 0$ then

$$\begin{aligned}
\frac{\sigma}{2} &< m(t_\varepsilon, \rho_\varepsilon) - M_{j_\varepsilon}^0 \\
&= m(t_\varepsilon, \rho_\varepsilon) - m(0, \rho_\varepsilon) + m(0, \rho_\varepsilon) - m(0, \xi_\varepsilon) + m(0, \xi_\varepsilon) - M_{j_\varepsilon}^0 \\
&\leq Lt_\varepsilon + L|\rho_\varepsilon - \xi_\varepsilon| = L|t_\varepsilon - n_\varepsilon| + L|\rho_\varepsilon - \xi_\varepsilon| \\
&\leq L\sqrt{2\|m_0\|_\infty}\,\varepsilon.
\end{aligned}$$

This is a contradiction if (5.15) holds. An analogous contradiction holds if $j_\varepsilon = 0$.

**Step 3. An inequality for $m$ via viscosity.** We check that

$$(t, \rho) \longmapsto m(t, \rho) - \phi(t, s_\varepsilon, \rho, \xi_\varepsilon) = m(t, \rho) - \psi(t, \rho)$$

has a maximum at $(t_\varepsilon, \rho_\varepsilon)$. Hence, since $m$ is a viscosity subsolution, we have

$$\frac{\partial \phi}{\partial t}(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) + H\left(\frac{\partial \phi}{\partial \rho}(t_\varepsilon, s_\varepsilon, \rho_\varepsilon, \xi_\varepsilon)\right) m(t_\varepsilon, \rho_\varepsilon) \leq 0.$$

Computing the derivatives

$$\frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + \lambda + H\left(\frac{2(\rho_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}\right) m(t_\varepsilon, \rho_\varepsilon) \leq 0. \tag{5.16}$$

**Step 4. An inequality for $M$ applying that $G$ is monotone.** As before, the function

$$(n, j) \longmapsto M_j^n - (-\phi(t_\varepsilon, h_t n, \rho_\varepsilon, h_\rho j)) = M_j^n - \psi(j, n)$$

has a minimum at $(n_\varepsilon, j_\varepsilon)$. In particular, we obtain

$$M_j^n \geq M_{j_\varepsilon}^{n_\varepsilon} - \psi(j_\varepsilon, n_\varepsilon) + \psi(j, n).$$

Since $G$ is monotone, it yields

$$M_{j_\varepsilon}^{n_\varepsilon} = G(M_{j_\varepsilon}^{n_\varepsilon - 1}, M_{j_\varepsilon - 1}^{n_\varepsilon - 1})$$

$$\geq G\left(\underbrace{M_{j_\varepsilon}^{n_\varepsilon} - \psi(j_\varepsilon, n_\varepsilon) + \psi(j_\varepsilon, n_\varepsilon - 1)}_{S_1}, \underbrace{M_{j_\varepsilon}^{n_\varepsilon} - \psi(j_\varepsilon, n_\varepsilon) + \psi(j_\varepsilon - 1, n_\varepsilon - 1)}_{S_2}\right).$$

Similarly to the proof of Theorem 5.5, for $h$ small, one can rewrite the previous inequality as

$$\frac{M_{j_\varepsilon}^{n_\varepsilon} - S_1}{h_t} + H\left(\frac{S_1 - S_2}{h_x}\right) M_{j_\varepsilon}^{n_\varepsilon} \geq 0.$$

Hence, we recover

$$\frac{\psi(j_\varepsilon, n_\varepsilon) - \psi(j_\varepsilon, n_\varepsilon - 1)}{h_t} + H\left(\frac{\psi(j_\varepsilon, n_\varepsilon - 1) - \psi(j_\varepsilon - 1, n_\varepsilon - 1)}{h_x}\right) M_{j_\varepsilon}^{n_\varepsilon} \geq 0. \tag{5.17}$$

We could compute this explicitly, but it is sufficient and clearer to apply the intermediate value theorem to deduce

$$-\frac{\partial \phi}{\partial s}(t_\varepsilon, \bar{s}_\varepsilon, \rho_\varepsilon, \xi_\varepsilon) + H\left(-\frac{\partial \phi}{\partial \xi}(t_\varepsilon, s_\varepsilon - h_t, \rho_\varepsilon, \bar{\xi}_\varepsilon)\right) M_{j_\varepsilon}^{n_\varepsilon} \geq 0$$

where $\bar{s}_\varepsilon \in (s_\varepsilon - h_t, s_\varepsilon)$ and $\bar{\xi}_\varepsilon \in (\xi_\varepsilon - h_\rho, \xi_\varepsilon)$. Hence, we conclude that

$$\frac{2(t_\varepsilon - \bar{s}_\varepsilon)}{\varepsilon^2} - \lambda + H\left(\frac{2(\rho_\varepsilon - \bar{\xi}_\varepsilon)}{\varepsilon^2}\right) M_{j_\varepsilon}^{n_\varepsilon} \geq 0. \tag{5.18}$$

**Step 5. An estimate for $\sigma$.** Substracting (5.16) from (5.18) we have that

$$\frac{\sigma}{4(1+T)} \leq \frac{s_\varepsilon - \bar{s}_\varepsilon}{\varepsilon^2} + H\left(\frac{2(\rho_\varepsilon - \bar{\xi}_\varepsilon)}{\varepsilon^2}\right) M_{j_\varepsilon}^{n_\varepsilon} - H\left(\frac{2(\rho_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}\right) m(t_\varepsilon, \rho_\varepsilon)$$

$$\leq \left(H\left(\frac{2(\rho_\varepsilon - \bar{\xi}_\varepsilon)}{\varepsilon^2}\right) - H\left(\frac{2(\rho_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}\right)\right) M_{j_\varepsilon}^{n_\varepsilon} + H\left(\frac{2(\rho_\varepsilon - \bar{\xi}_\varepsilon)}{\varepsilon^2}\right) (M_{j_\varepsilon}^{n_\varepsilon} - m(t_\varepsilon, \rho_\varepsilon))$$

Notice that the second term is non-positive due to (5.14). We now use the Lipschitz continuity of $H$, which holds for the cut-off given by (5.6), and we obtain that

$$\frac{\sigma}{8} \leq C\left|\frac{2(\bar{\xi}_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}\right| \leq C\frac{h_\rho}{\varepsilon^2}\|m_0\|_{L^\infty}.$$

**Step 6. A first choice of** $\varepsilon$. We take $\varepsilon = C\sigma$ where $C$ is chosen so that (5.15) hold. Then, we have that

$$\sigma^3 \le Ch_\rho.$$

where $C(T, \alpha)$ is independent of $h_t$, $h_x$ or $\delta$. This completes the proof. $\qquad\square$

**Remark 5.8.** Notice that we do not use the equation until step 4 and that the Lipschitz continuity of $m_0$ plays a key role. However, the homogeneous boundary conditions do not.

**Remark 5.9.** Notice that we recover the exponent $h^{\frac{1}{3}}$ from the Lipschitz continuity of $H$. If $H$ is only $\alpha$-Hölder continuous as in [9], then the rate of convergence is given by $h^{\frac{\alpha}{1+2\alpha}}$.

# 6 Numerical results

## 6.1 Asymptotics as $t \to +\infty$

Through numerical experiments, we see that the vortex seems to be the asymptotic solution also in $u$ variable. In Figure 7 we represent the asymptotic state of the two-bump initial data constructed explicitly for small times in Section 4.4. We recall that why the computations in Section 4.4 are only valid for small time is that the first bump reaches the second bump, and we did not compute the first shock after this happens. However, as we see in Figure 7, the first bump "eats" the second bump (possibly in finite time), and we recover the vortex. Since $u_0(0) = 0$, we have that $u(t, 0) = 0$ so the vortex cannot be reached in the supremum norm. Notice also that if $u_0(0) \ne 0$, then $u(0, t) = (u_0(0)^{-\alpha} + \alpha t)^{-\frac{1}{\alpha}}$. Nevertheless, the simulation suggest convergence in all $L^p$ norms for $1 \le p < \infty$.
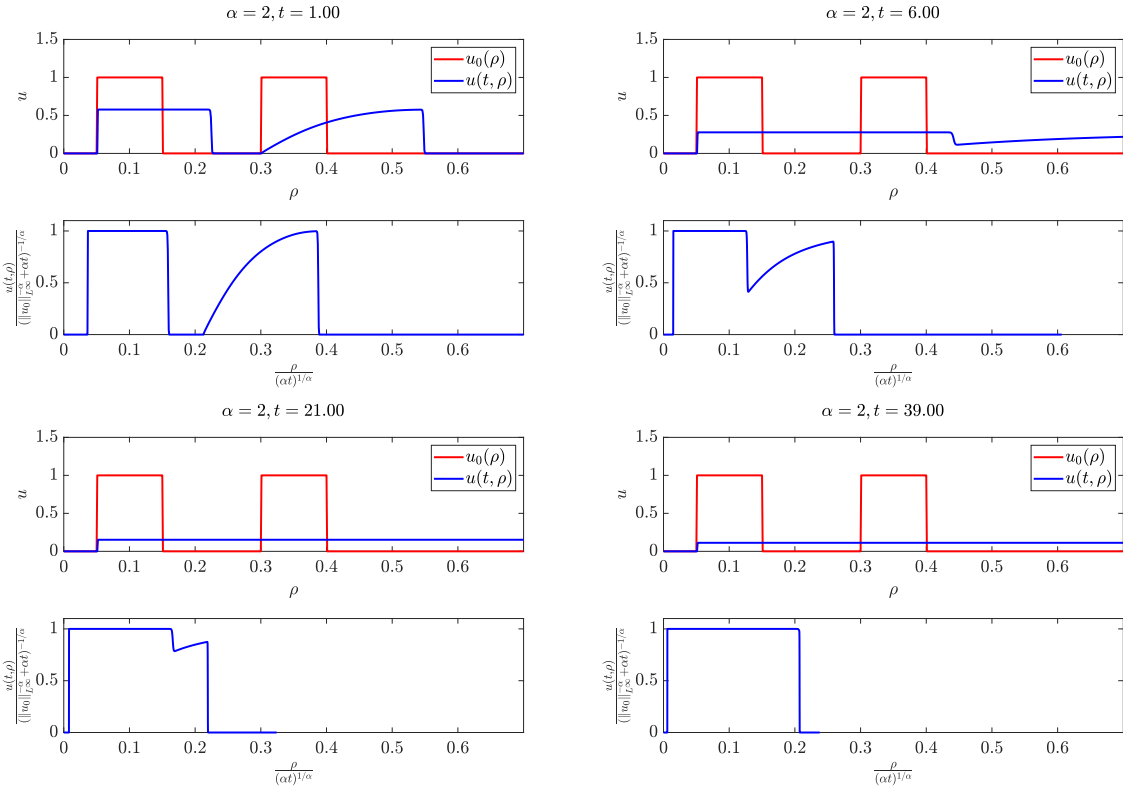


Figure 7: Asymptotic behaviour of $u$ in rescaled variables. See a movie simulation in the supplementary material [20, Video 3].

It is an open problem to determine if the first singularity catches the boundary front in finite time for these particular solutions.

## 6.2 Comparison of the waiting time

It is interesting to compare the behaviour of different powers $u_0(\rho) = (\beta + 1)(1 - \rho)_+^{\beta}$ which have total mass $M = 1$. There is waiting time if $\beta \geq \frac{1}{\alpha - 1}$ (see Corollary 4.9 and Theorem 4.10). We will work with $\alpha = 2$. Since the masses are ordered, the waiting time for $u_0 = 2(1 - \rho)_+$ is shorter than that of $u_0 = 3(1 - \rho)_+^2$. It is interesting to notice that the solution for $u_0 = 3(1 - \rho)_+^2$ develops a singularity at the interior of the support, before the support starts moving.
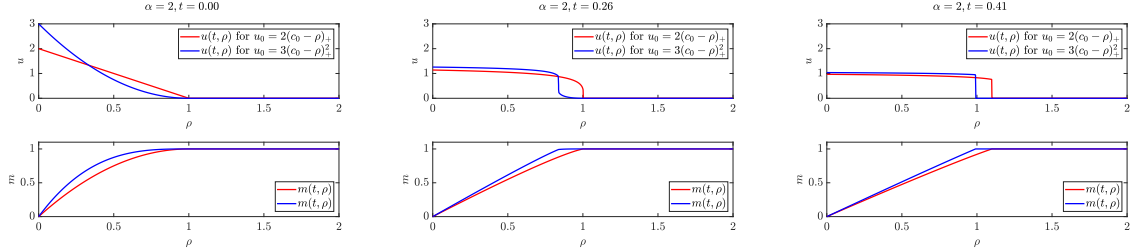


Figure 8: Behaviour of two different powers with waiting time. See a movie simulation in the supplementary material [20, Video 4].

## 6.3 Level sets of a solution with and without waiting time

In Section 4.6 we showed that $t^{\frac{1}{\alpha}}$ is the asymptotic behaviour of the support of $u$ for compactly supported $u_0$. For instance, if $u_0$ is a Dirac $\delta$ function at $S(0)$ of mass $M$, we have shown that the support is $[S(0), S(0) + M(\alpha t)^{\frac{1}{\alpha}}]$. However, for solutions with waiting time, we do not know what is the behaviour of the support for $t$ small. We illustrate an example when $u_0 = (1 - \rho)_+$ for $\alpha = 2$ in Figure 9 (cf. Figure 6). This initial datum produces a solution with waiting time due to Corollary 4.9, which by Theorem 3.4 is initially given by the generalised characteristics. However, as pointed out in Remark 3.1 the characteristics are not the level sets of $m$. Notice that the level sets of $m$ are not straight even for $t$ small. For comparison, we show a solution not given by characteristics (therefore not a classical solution) and without waiting time (by Theorem 4.10) which we represent in Figure 10.
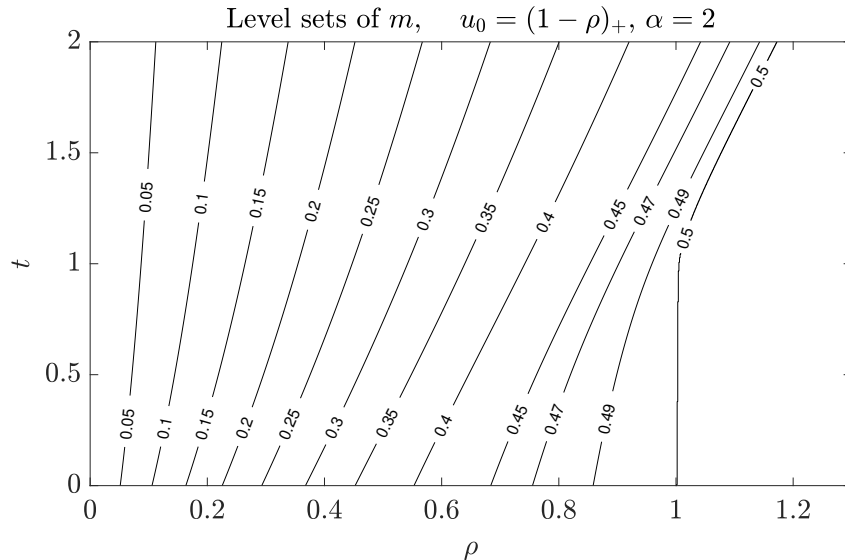


Figure 9: Level sets of the numerical solution with $u_0 = (1 - \rho)_+$ for $\alpha = 2$, and a uniform mesh in space of equispaced grid $h_\rho = 1e - 3$. In Figure 6 the reader may find a comparison with the mass subsolution with explicit Ansatz.
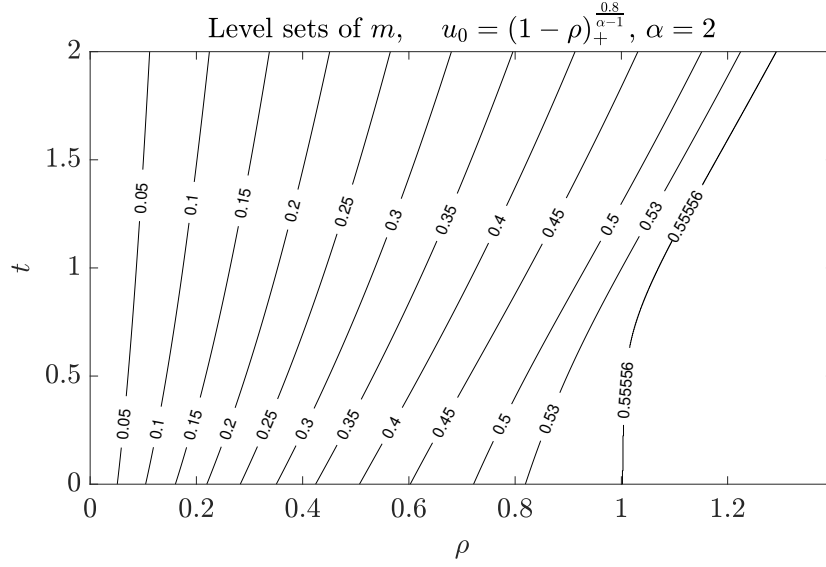
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Figure 10: Level sets of the numerical solution not given by characteristics presented in Remark 3.8, which does not have waiting time due to Theorem 4.10.

## Remarks and open problems

1. We have constructed a theory of radial solutions and proved well-posedness of the mass formulation. Uniqueness in terms of the $u$ variable is an open problem.

2. Is there a non-radial theory? This seems to be a very difficult problem.

3. Is there asymptotic convergence to the vortex solution in the $u$ variable in general?

4. An interesting problem is to construct a theory for infinite mass solutions.

5. In the two bump solution, is there actually convergence to the vortex in finite time? The numerical experiments suggest so. The ODE for $S_1$ can be written explicitly from the Rankine-Hugoniot condition, and the question is whether $S_1(t) = S_2(t)$ for some $t > 0$.

## Acknowledgments

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