Compensated Integrability and Conservation Laws

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Plan of the talk

What are the objects under consideration ?
Div-BV or Div-free symmetric tensors.

Why are they ubiquitous ?
Because of Nœther's Theorem.

How do we treat them ?
With Compensated Integrability.

Which results do we get ?
Dispersion (Strichartz-like) estimates.

WHAT ? Div-free tensors

$U \subset \mathbb{R}^n$ an open domain.

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Definition 2

The tensor A is Div-free if its entries are Radon measures, and $\mathrm{Div}\, A \equiv 0$.

Two examples

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If $n = 2$, every Div-free tensor is special.

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Equivalently, $A : U \to \mathbf{Sym}_n$ is Div-BV if its entries and the coordinates $(\operatorname{Div} A)_j$, **and** its (well-defined) normal trace $A\vec{\nu}$ are finite measures.

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$$\begin{aligned}\partial_t\rho + \mathrm{div}_y(\rho u) &= 0,\\ \partial_t(\rho u) + \mathrm{Div}_y(\rho u\otimes u) &= \mathrm{Div}_y\,\Sigma,\end{aligned}$$

where $\Sigma(t,y)\in\mathbf{Sym}_d$ is the stress tensor.

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This can be recast as $\mathrm{Div}_{t,y} A = 0$ with

$$A = \begin{pmatrix} \rho & \rho u^T \\ \rho u & \rho u \otimes u - \Sigma \end{pmatrix} \in \mathbf{Sym}_{1+d}.$$

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The (symmetric !) energy-momentum tensor

$$A = \begin{pmatrix} \frac{\rho c^2 + p}{c^2 - |v|^2} - \frac{p}{c^2} & \frac{\rho c^2 + p}{c^2 - |v|^2}\, v \\ \frac{\rho c^2 + p}{c^2 - |v|^2}\, v & \frac{\rho c^2 + p}{c^2 - |v|^2}\, v \otimes v + pI_3 \end{pmatrix}$$

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involves admissible variations that are not additive ($\alpha_\epsilon \neq \alpha + \epsilon\beta$), but *are* "compositional" :

$$\alpha_\epsilon = \underbrace{\phi_\epsilon^* \alpha}_{\text{pullback}}, \qquad (\phi_\epsilon)_{\epsilon \in \mathbb{R}} \text{ a flow in } U.$$

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Other invariances $\longrightarrow$ If G contains $O(q)$ for a non-degenerate quadratic form $q(x) = x^T Sx$ ($q(x) = |x|^2$ or $q(t,y) = c^2t^2 - |y|^2$), then $A := S^{-1}T$ is symmetric and still Div-free.

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Its closedness expresses the Gauß–Faraday law

$$\partial_t \vec{B} + \mathrm{curl}\, \vec{E} = 0, \qquad \mathrm{div}\, \vec{B} = 0.$$

The Div-free energy-momentum tensor :

$$A = \begin{pmatrix} L - \vec{E} \cdot \vec{D} & \vec{H} \times \vec{E} \\ \vec{D} \times \vec{B} & (L + \vec{B} \cdot \vec{H})\, I_3 - \vec{E} \otimes \vec{D} - \vec{H} \otimes \vec{B} \end{pmatrix},$$

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which is equivalent to Lorentz invariance :

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But ∇ is elliptic, while Div is not !! In the spirit of Compensated Compactness, we expect that some non-linear quantity $D(A)$ behaves better than the entries a_{ij} do individually ...

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For special tensors, this means θ convex.

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Observe that $(\det A)^{\frac{1}{n}}$ is a well-defined measure,

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Dual structure : the "2nd" BVP for the Monge-Ampère equation

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The proof exploits Brenier's theorem in Optimal Transport.

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$$\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(B_n)} \le \left(\frac{\mathrm{Area}(\partial\Omega)}{\mathrm{Area}(\partial B_n)}\right)^{\frac{n}{n-1}}.$$
- With $A = f(x) I_n$, one recovers the Sobolev embedding
$$BV(\mathbb{R}^n) \subset L^{\frac{n}{n-1}}(\mathbb{R}^n).$$

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Looks like Jensen's Inequality ...

Periodic framework

Theorem 3 (D.S., Ann. IHP 2018.)

Let $A \succ 0_n$ be a periodic Div-free tensor. Then

$$\fint (\det A)^{\frac{1}{n-1}}\,dx \le \left(\det \fint A\,dx\right)^{\frac{1}{n-1}}.$$

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but $\det^{\frac{1}{n-1}}$ is **not** concave over $\mathbf{Sym}_n^+$.

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Theorem 4 (L. de Rosa & R. Tione, JFA 2020.)

Let $A_m \succ 0_n$ *be a sequence of Div-BV tensors, such that* $\mathrm{Div}\, A_m$ *is bounded in* $\mathcal{M}(U)$ *and* $A_m \stackrel{*}{\rightharpoonup} A$ *in* L^p *with* $p > \frac{n}{n-1}$ *. Then up to a subsequence*

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See also

- Skipper & Wiedemann (2021),
- Guerra, Raiță & Schrecker (2021, 2022).

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Theorem 5 (D.S. 2021.)

Let $A \succ 0_n$ be a Div-free tensor over $(0, T) \times \mathbb{R}^d$. Then

$$\int_0^T dt \int_{\mathbb{R}^d} (\rho \det \sigma)^{\frac{1}{d}}\, dx \leq_d M^{\frac{1}{d}} \left(\|m(0, \cdot)\|_{\mathcal{M}} + \|m(T, \cdot)\|_{\mathcal{M}} \right).$$

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Hypotheses : physical domain $\mathbb{R}^d$*, finite mass* M *and initial energy* E_0*.*

Euler system of GD

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$$MI_0 \quad \longmapsto \quad \frac{1}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\rho_0(y)\rho_0(z)|z-y|^2\,dz\,dy.$$

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- (Mono-atomic) Compare

$$\int_0^{+\infty} t\,dt \int_{\mathbb{R}^d} \rho^{\frac{1}{d}} p\,dy \leq_d M^{\frac{1}{d}} \sqrt{MI_0}$$

with (J.-Y. Chemin, 1990)

$$t^2 \int_{\mathbb{R}^d} p\,dy \leq \frac{2}{d}\, I_0.$$

Kinetic equations (Boltzman)

$f(t, y, \xi)$ the distribution of mass, ξ the velocity of particles.

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Ya. Sinai's question :

> *Is the number K of collisions finite ? If so, how does it behave with N ?*

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Mind that K may be exponentially large !

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Related to Minkowski's problem, solved by Pogorelov (1978).

Theorem 7 (D.S., ARMA 2021.)

Then

$$\sum_{\text{coll.}} \underbrace{|u_{\text{out}} - u_{\text{in}}|}_{\text{weight}} \leq_d N^2 \bar{u}.$$

Way better than N^N ... !

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In other words ($q_\alpha = m u_\alpha$ the linear momenta)

$$\text{mean}\,[TV(t \mapsto q_\alpha(t))] \leq_d \sqrt{ME_0}\,.$$

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$$\partial_t u + \mathrm{div}_y \vec{f}(u) = 0.$$

Multi-D scalar conservation laws

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Analysis based upon the tensor $A = T \circ u$,

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A is Div-BV.

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$$\partial_t u + \partial_1 \frac{u^2}{2} + \cdots + \partial_d \frac{u^{d+1}}{d+1} = 0.$$

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Theorem 8 (D. S. & L. Silvestre, ARMA 2019.)

There are exponents $\alpha(d,p), \beta(d,p) > 0$ *such that the solution of the Cauchy problem with* $u_0 \in L^1(\mathbb{R}^d)$ *exists and satisfies for every* $1 < p \leq +\infty$ *and every* $t > 0$

$$\|u(t)\|_p \leq_{d,p} \|u_0\|_1^{\alpha} t^{-\beta}.$$

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Corollary 1 (M. Crandall's question (1972).)

The PDE and the entropy inequalities are satisfied in the distributional sense.

Thank you for your attention !