COMPACTNESS AND THE CURVATURE OF 3-WEBS

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I. Weak convergence and products

Assume

$$\begin{cases} u^\epsilon \rightharpoonup u \\ v^\epsilon \rightharpoonup v \end{cases} \qquad \text{as } \epsilon \to 0.$$

QUESTION

When is it true that

$$\boxed{u^\epsilon v^\epsilon \rightharpoonup uv} \quad ??$$

FALSE IN GENERAL: High frequencies in u^ϵ and v^ϵ may "resonate".

EXAMPLE:

$$u^\epsilon = v^\epsilon = \sin(\tfrac{x}{\epsilon}) \rightharpoonup 0, \;\; u^\epsilon v^\epsilon = \sin^2(\tfrac{x}{\epsilon}) \rightharpoonup \tfrac{1}{2}.$$

THEOREM

Let $n = 2$ and assume

$$u_t^\epsilon + b(x,t)u_x^\epsilon = 0, \;\; v_t^\epsilon + c(x,t)v_x^\epsilon = 0.$$

If

$$b \neq c,$$

then

$$u^\epsilon v^\epsilon \rightharpoonup uv.$$

PROOF: (L. Tartar) Define

$$U^\epsilon = [bu^\epsilon, u^\epsilon]^T, V^\epsilon = [v^\epsilon, -cv^\epsilon]^T.$$

We have

$$\begin{cases} \mathrm{div}\, U^\epsilon = (bu^\epsilon)_x + u_t^\epsilon = b_x u^\epsilon \\ \text{curl } V^\epsilon = v_t^\epsilon - (-cv^\epsilon)_x = c_x v^\epsilon. \end{cases}$$

By Div-Curl Lemma,

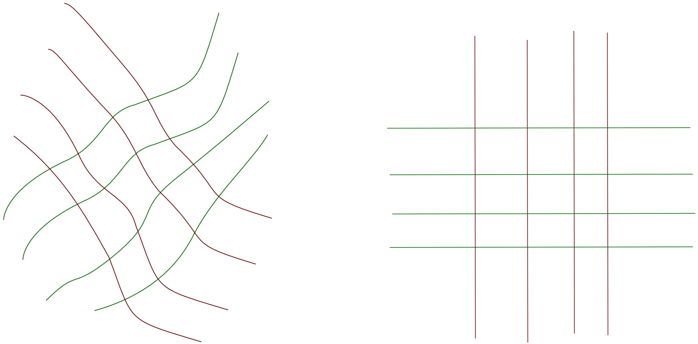
$$U^\epsilon \cdot V^\epsilon \rightharpoonup U \cdot V.$$

This says

$$(b-c)u^\epsilon v^\epsilon \rightharpoonup (b-c)uv.$$

□

ANOTHER PROOF: Change variables:



Convert to $u^\epsilon = u^\epsilon(x)$, $v^\epsilon = v^\epsilon(t)$. Easy to see that

$$u^\epsilon(x)v^\epsilon(t) \rightharpoonup u(x)v(t).$$

II. Weak convergence and triple products

Assume

$$\begin{cases} u^\epsilon \rightharpoonup u \\ v^\epsilon \rightharpoonup v \\ w^\epsilon \rightharpoonup w \end{cases} \qquad \text{as } \epsilon \to 0.$$

QUESTION

When is it true that

$$\boxed{u^\epsilon v^\epsilon w^\epsilon \rightharpoonup uvw} \quad ??$$

REFERENCES: J.-L. Joly, G. Metivier and J. Rauch, "Trilinear compensated compactness and nonlinear geometric optics", Annals of Math. 142 (1995), 121–169.

M. Christ, "On trilinear oscillatory integral inequalities and related topics", preprint (2021)

THEOREM (Joly–Metivier–Rauch)

Assume

$$u^\epsilon = u^\epsilon(x),\ \ v^\epsilon = v^\epsilon(t),\ \ w^\epsilon_t + a(x,t)w^\epsilon_x = 0.$$

If $a > 0$ *and*

$$(\log a)_{xt} \neq 0,$$

then

$$u^\epsilon v^\epsilon w^\epsilon \rightharpoonup uvw.$$

WHAT IS THE MEANING OF THE CONDITION

$$\boxed{\kappa = (\log a)_{xt} \neq 0}\ ?$$

This is a formula for the curvature of the 3-web comprising the horizontal lines, the vertical lines and the trajectories of the ODE

$$\dot\gamma = a(\gamma, t).$$

Introduce the simple transport PDE

$$\boxed{\phi_t + a\phi_x = 0} \quad \text{in } \mathbb{R}^2,$$

any solution $\phi = \phi(x,t)$ of which is constant along the flow lines of the ODE $\dot{\gamma} = a(\gamma, t)$.

LEMMA

Assume

$$\phi_x > 0,\ \ \phi_t < 0$$

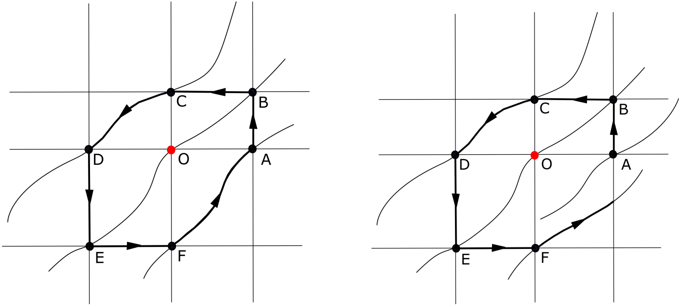
and define

$$z = \frac{\phi_{xt}}{\phi_t}.$$

Then

$$z_t + (az)_x = \kappa.$$

III. 3-webs in the plane



THEOREM

(i) *If $\kappa \equiv 0$, then the points A, B, C, D, E, F are the vertices of a closed curvilinear "hexagon".*

(ii) *If instead $\kappa \neq 0$, the points A, B, C, D, E, F are not the vertices of a closed hexagon.*

A PDE PROOF: 1. Assume $O = (0,0)$ and let ϕ solve the transport PDE with initial conditions

$$\phi(x,0) = \int_0^x \frac{1}{a(y,0)}\, dy.$$

Then

$$\phi_t(x,0) = -a(x,0)\phi_x(x,0) = -a(x,0)\frac{1}{a(x,0)} = -1,$$

and so $\phi_{xt}(x,0) = 0$. Thus

$$z(x,0) = 0,$$

where $z = \frac{\phi_{xt}}{\phi_t}$.

2. Assume $\kappa \equiv 0$. It follows from the Lemma that $z \equiv 0$. Therefore $\phi_{xt} \equiv 0$ and consequently

$$0 = \iint_{OABC} \phi_{xt}\, dxdt = \phi(B) + \phi(O) - \phi(A) - \phi(C),$$

$$0 = \iint_{ODEF} \phi_{xt}\, dxdt = \phi(E) + \phi(O) - \phi(D) - \phi(F).$$

Since $\phi(O) = \phi(B) = \phi(E) = 0$ and $\phi(C) = \phi(D)$, it follows that $\phi(F) = \phi(A)$. So the points A and F are on the same flow line.

3. Suppose instead that $\kappa < 0$. Since $z_t + (az)_x = \kappa$ in $\mathbb{R}^2$, with $z = 0$ on the horizontal line $\{t = 0\}$, we have

$$\begin{cases} z < 0 & \text{in } \mathbb{R} \times \{t > 0\} \\ z > 0 & \text{in } \mathbb{R} \times \{t < 0\}. \end{cases}$$

As $\phi_{xt} = \phi_t z$ and $\phi_t < 0$,

$$\begin{cases} \phi_{xt} > 0 & \text{in } \mathbb{R} \times \{t > 0\} \\ \phi_{xt} < 0 & \text{in } \mathbb{R} \times \{t < 0\}. \end{cases}$$

Therefore

$$0 < \iint_{OABC} \phi_{xt}\, dxdt = \phi(B) + \phi(O) - \phi(A) - \phi(C),$$

$$0 > \iint_{ODEF} \phi_{xt}\, dxdt = \phi(E) + \phi(O) - \phi(D) - \phi(F).$$

Since $\phi(O) = \phi(B) = \phi(E) = 0$ and $\phi(C) = \phi(D)$, we have $\phi(A) < \phi(F)$. So A and F are not on the same flow line: the hexagon does not close up. □

Zero curvature gives resonances

It turns out that if $\kappa \equiv 0$ (equivalently, all the hexagons close up), then there exist 3 functions

$$\psi_1(x),\ \psi_2(t),\ \psi_3(x,t),$$

with non vanishing gradients, such that $\psi_{3,t} + a\psi_{3,x} = 0$ and

$$\psi_1(x) + \psi_2(t) + \psi_3(x,t) \equiv 0 \quad \text{(Resonance condition)}$$

(Proof: Solve the PDE for ψ_3 by separating variables.)

Let

$$u^\epsilon(x) = e^{i\frac{\psi_1(x)}{\epsilon}}, v^\epsilon(t) = e^{i\frac{\psi_2(t)}{\epsilon}}, w^\epsilon(x,t) = e^{i\frac{\psi_3(x,t)}{\epsilon}}.$$

Then

$$u^\epsilon, v^\epsilon, w^\epsilon \rightharpoonup 0$$

by (non)stationary phase estimates, but

$$u^\epsilon v^\epsilon w^\epsilon \equiv 1.$$

IV. Compactness and curvature

Assume $\{u^\epsilon(x)\}$ is bounded in $L^\infty(\mathbb{R})$, $w^\epsilon = w^\epsilon(x,t)$ solves the PDE

$$\boxed{w^\epsilon_t + a(x,t)w^\epsilon_x = 0,}$$

and

$$w^\epsilon \rightharpoonup 0 \quad \text{as } \epsilon \to 0.$$

Let $\chi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth cutoff function and introduce the nonlinear correlation function

$$\boxed{\lambda^\epsilon(t) = \int_{\mathbb{R}} u^\epsilon(x) w^\epsilon(x,t) \chi(x,t)\, dx.}$$

THEOREM

Assume that

$$\kappa \neq 0 \quad \text{in } \mathbb{R}^2.$$

Then

$$\lambda^\epsilon \to 0 \quad \text{strongly in } L^2(\mathbb{R}^2).$$

OUTLINE OF PROOF:

1. Write

$$I^\epsilon = \int_0^T (\lambda^\epsilon)^2\,dt = \int_0^T \iint_{\mathbb{R}^2} u^\epsilon(x)u^\epsilon(y)w^\epsilon(x,t)w^\epsilon(y,t)\chi(x,t)\chi(y,t)\,dxdydt.$$

We have

$$w^\epsilon(x,t) = v^\epsilon(\phi(x,t)),$$

where ϕ solves the transport PDE and $v^\epsilon \rightharpoonup 0$. Also,

$$v^\epsilon(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi z}\widehat{v^\epsilon}(\xi)\,d\xi,$$

where

$$\widehat{v^\epsilon} \to 0 \quad \text{uniformly on bounded sets.}$$

Hence

$$I^\epsilon = \iiiint_{\mathbb{R}^4} \Lambda^\epsilon J_1\,dxdyd\xi d\eta,$$

for

$$\Lambda^\epsilon = u^\epsilon(x)u^\epsilon(y)\widehat{v^\epsilon}(\xi)(\widehat{v^\epsilon}(\eta))^-,$$

$$J_1 = \int_0^T e^{i(\xi\phi(x,t)-\eta\phi(y,t))}b_1\,dt.$$

2. Assume that for each (x, y, ξ, η), the mapping

$$t \mapsto \xi\phi(x, t) - \eta\phi(y, t)$$

has a unique, nondegenerate minimum at $\tau = \tau(x, y, \xi, \eta)$. Then standard stationary phase estimates show

$$I^\epsilon = \iiiint_{\mathbb{R}^4} \Lambda^\epsilon J_2 \, dxdyd\xi d\eta + o(1),$$

where

$$J_2 = e^{i\Psi(x,y,\xi,\eta)} b_2 |\xi|^{-\frac{1}{2}}$$

for

$$\Psi(x, y, \xi, \eta) = \xi\phi(x, \tau) - \eta\phi(y, \tau), \ \tau = \tau(x, y, \xi, \eta).$$

3. Now define the Fourier integral operator

$$\mathcal{T}f(x, y) = \iint_{\mathbb{R}^2} e^{i\Psi(x,y,\xi,\eta)} b_2 f(\xi, \eta) \, d\xi d\eta.$$

I claim

$$\mathcal{T} : L^2(\mathbb{R}^2_{\xi\eta}) \to L^2(\mathbb{R}^2_{xy})$$

is a bounded linear operator.

The key observation for showing this is that

$$\det \begin{pmatrix} \Psi_{x\xi} & \Psi_{x\eta} \\ \Psi_{y\xi} & \Psi_{y\eta} \end{pmatrix} = B \int_x^y \kappa(r,\tau)\, dr \neq 0,$$

where B denotes a nonvanishing expression.

Since $\mathcal{T}$ is a bounded linear operator on L^2, the extra term $|\xi|^{-\frac{1}{2}}$ above lets us show that

$$I^\epsilon \to 0.$$

□

REMARK In the real proof, we have to factor

$$w^\epsilon(x,t) = v^\epsilon(\phi(x,t)) = \tilde{v}^\epsilon(\tilde{\phi}(x,t))$$

for two different solutions of the transport PDE, to get to the situation stated in blue on the previous slide.