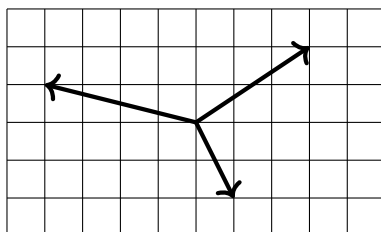


Geometry – Solutions

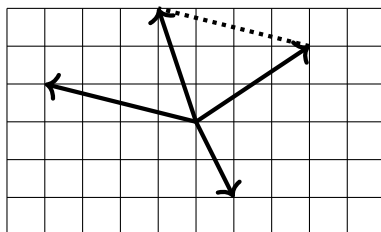
Revision Questions

1. Something like



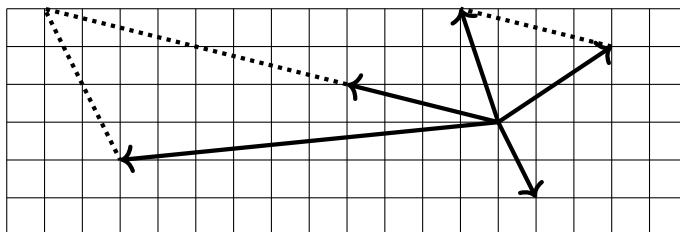
2. We add the components separately, so $\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

My diagram now looks like this.



3. You multiply a vector by a scalar by multiplying each component, so $3\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix}$ and $2\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$. Then add them together $\begin{pmatrix} -12 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -10 \\ -1 \end{pmatrix}$.

My diagram now looks like this.



4. This line has gradient $(-1-5)/(3-1) = -3$ and goes through $(1, 5)$ so it's $y-5 = -3(x-1)$ which can also be written as $y = 8 - 3x$.
5. This must be $y = 2x + c$ for some constant c , and the line goes through $(3, 5)$ so $5 = 6 + c$ and so the line is $y = 2x - 1$.
6. I might try to show that all the sides are the same length, and that all the corners are right angles. First I need to draw a diagram to get the points in the right order.

$$\begin{array}{c} (2, 6) \bullet \\ (0, 5) \bullet \quad (3, 4) \bullet \\ (1, 3) \bullet \end{array}$$

Now I can check that the distances from $(1, 3)$ to $(3, 4)$, from $(3, 4)$ to $(2, 6)$, from $(2, 6)$ to $(0, 5)$, and from $(0, 5)$ to $(1, 3)$ are all $\sqrt{5}$.

To check the corners are right angles, I could check that the gradients of the lines for each side multiply to -1 . Those gradients are all either $\frac{1}{2}$ or -2 , so all the corners are right angles.

7. There are lots of examples that work! I decided to use the x -axis as one of my lines (that's $y = 0$), and then use something like $y = \sqrt{3} - ax$ and $y = \sqrt{3} + bx$ for some a and b ; I've chosen those y -intercepts so that $(0, \sqrt{3})$ is a corner of the triangle.

I need those two lines to go through $(\pm 1, 0)$. I can do that by choosing a and b carefully, and I end up with the three lines $y = 0$ and $y = \sqrt{3}(1 - x)$ and $y = \sqrt{3}(1 + x)$.

8. $(x + 1)^2 + (y - 2)^2 = 3^2$

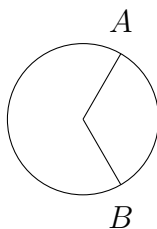
The area is πr^2 and $r = 3$ so the area is 9π .

The circle meets the x -axis where $(x + 1)^2 + (0 - 2)^2 = 3^2$. That's $x = -1 \pm \sqrt{5}$.

The circle meets the y -axis where $(0 + 1)^2 + (y - 2)^2 = 3^2$. That's $y = 2 \pm \sqrt{8}$.

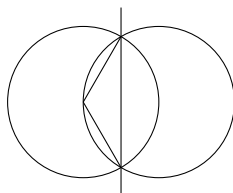
9. $x^2 + 9x + y^2 - 3y = (x + \frac{9}{2})^2 + (y - \frac{3}{2})^2 - \frac{81}{4} - \frac{9}{4}$. The equation of the circle is $(x + \frac{9}{2})^2 + (y - \frac{3}{2})^2 = 10 + \frac{90}{4}$. So the centre is $(-\frac{9}{2}, \frac{3}{2})$ and the radius is $\sqrt{\frac{65}{2}}$.

10. Draw a diagram.



Since 120° is one-third of 360° , the length of the arc is one-third of the length of the circumference $2\pi r$ with $r = 2$. So the length of the arc is $\frac{4}{3}\pi$. The area is one-third of πr^2 , which works out to be $\frac{4}{3}\pi$.

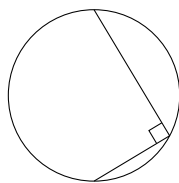
11. Draw a diagram.



Find the points of intersection. Taking the difference between the two equations gives $x^2 = (x - 2)^2$, so $x = 2 - x$ or $x = x - 2$, which only has $x = 1$ as a solution. The y -coordinates are $\pm\sqrt{3}$, and the angle at the centre is 120° . Let's aim to find the area to the right of $x = 1$ that's inside both circles. That's the area of the sector from the previous question, minus the area of a triangle. We can use $\frac{1}{2}ab \sin \theta$ to work out the area of the triangle, $\sqrt{3}$.

Then we'll need to double the area to get our final answer of $\frac{8}{3}\pi - 2\sqrt{3}$.

12. Draw a diagram.



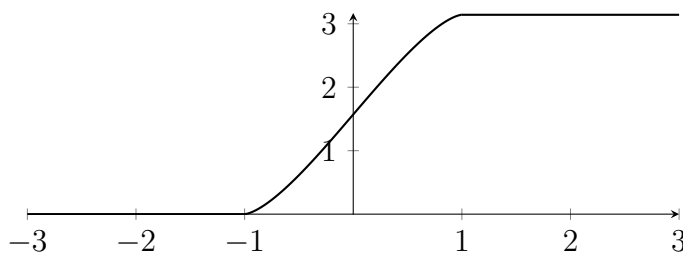
We could write down equations for the distance of a general point (x, y) to each of these points and set them equal to each other, but that's a lot of work.

Instead, note that the gradient of the line from $(0, 0)$ to $(1, a)$ is a and the gradient of the line from $(1, a)$ to $(0, a + a^{-1})$ is $-a^{-1}$. These gradients multiply to -1 , so the lines are at right-angles.

The angle in a semi-circle is a right-angle, so the line from the first point to the third point is the diameter of the circle.

The centre is at the midpoint of the diameter, so it's at $(0, \frac{1}{2}(a + a^{-1}))$.

13. The area $A(c)$ is zero if $c < -1$ and it's π if $c > 1$. In between, the area rises from 0 to π in a nice symmetric manner; slow then fast then slow.



MAT Questions**MAT 2016 Q1C**

- The equation can be written as

$$\left(x + \frac{a}{2}\right) + \left(y + \frac{a}{2}\right) = c + \frac{a^2}{4} + \frac{b^2}{4}.$$

- The centre is at $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and the radius is $\sqrt{c + \frac{a^2}{4} + \frac{b^2}{4}}$.
- The origin is in the circle if the distance to the centre is less than the radius
- This gives the inequality $\sqrt{\frac{a^2}{4} + \frac{b^2}{4}} < \sqrt{c + \frac{a^2}{4} + \frac{b^2}{4}}$. Both sides are positive, so we want $\frac{a^2}{4} + \frac{b^2}{4} < c + \frac{a^2}{4} + \frac{b^2}{4}$, which happens only if $c > 0$.
- Alternatively, substitute in $x = 0$ and $y = 0$ to discover that the origin lies on the boundary of the circle when $c = 0$. The origin is outside the circle if c is very small, so the answer must be $c > 0$.
- The answer is (a).

MAT 2017 Q1G

- No matter the value of θ , the point $(-1, 1)$ is always on the line. As θ changes, the line rotates around, such that the line makes an angle of θ with the positive x -axis.
- The “larger” region is maximised if the line is at right angles to the radius between $(0, 0)$ and $(-1, 1)$.
- This happens for two values of θ , 45° and 225° . (there are two values of θ corresponding to the same line, because replacing θ with $\theta + 180^\circ$ introduces a minus sign on *both* sides of the equation).
- The answer is (b).

MAT 2014 Q1D

- The line perpendicular to $y = mx$ that passes through $(1, 0)$ is $y = -\frac{1}{m}(x - 1)$.
- This line meets $y = mx$ when $x = \frac{1}{1 + m^2}$ and so $y = \frac{m}{1 + m^2}$.
- We want the point on the “other side” of the line. Moving an equal distance will change the y -coordinate by the same amount again, that is, doubling it. So the point we’re looking for has $y = \frac{2m}{1 + m^2}$.
- The answer is (d).

MAT 2008 Q4

- (i) Let's write down $(x - a)^2 + (y - b)^2 = r^2$ for the equation of the circle. The three points $(0, 0)$ and $(p, 0)$ and $(0, q)$ all lie on the circle, which gives three equations;

$$(0 - a)^2 + (0 - b)^2 = r^2, \quad (p - a)^2 + (0 - b)^2 = r^2, \quad (0 - a)^2 + (q - b)^2 = r^2.$$

We can simplify these a bit, and multiply out some squares, and take the difference between pairs of equations, to get

$$p^2 - 2pa = 0, \quad q^2 - 2qb = 0, \quad a^2 + b^2 = r^2$$

Now $p \neq 0$ and $q \neq 0$ so $a = \frac{p}{2}$ and $b = \frac{q}{2}$, and $r = \sqrt{\frac{p^2 + q^2}{4}}$.

If we substitute those numbers into the equation $(x - a)^2 + (y - b)^2 = r^2$ and multiply out the squares, we get the equation in the question.

Along the way, we found the centre $(a, b) = (\frac{p}{2}, \frac{q}{2})$ and we found the radius. The area of C is $\pi \frac{p^2 + q^2}{4}$.

- (ii) We just found the area of the circle. The area of the triangle is $\frac{1}{2}pq$. If we write out the inequality in the question, we see that we're being asked to prove that

$$\pi \frac{p^2 + q^2}{2pq} \geq \pi$$

for all positive real numbers p and q . This is true because $(p - q)^2 \geq 0$, and that rearranges to the inequality above (we can divide by pq because p and q are not zero). If I'm honest, I rearranged the equation first, factorised it as $(p - q)^2$, realised that was positive or zero, then presented all of that to you in the opposite order. Sometimes it's a good idea to work backwards as well as forwards... provided that your final argument makes sense, of course.

- (iii) Now we're asked to solve

$$\pi \frac{p^2 + q^2}{2pq} = 2\pi$$

which rearranges to $p^2 + q^2 = 4pq$. This is one equation for two variables, so we can't really solve it for p and q . But we just want expressions for the angles. From trigonometry, we know that q/p is $\tan \angle OPQ$, and something similar is true for $\tan \angle OQP$. That inspires me to divide the equation by p^2 and solve

$$1 + \left(\frac{q}{p}\right)^2 = 4\left(\frac{q}{p}\right)$$

which is just a quadratic equation. The roots are $2 \pm \sqrt{3}$ so the angles are $\tan^{-1}(2 \pm \sqrt{3})$.

Extension

- If $\tan \theta = 2 - \sqrt{3}$ then

$$\tan 2\theta = \frac{4 - 2\sqrt{3}}{1 - (4 - 4\sqrt{3} + 3)} = \frac{4 - 2\sqrt{3}}{4\sqrt{3} - 6} = \frac{1}{\sqrt{3}}$$

so 2θ is 30° (restricting to the range $0 \leq \theta \leq 180^\circ$). So θ must be 15° .

If $\tan \theta = 2 + \sqrt{3}$ then

$$\tan 2\theta = \frac{4 + 2\sqrt{3}}{1 - (4 + 4\sqrt{3} + 3)} = \frac{4 + 2\sqrt{3}}{-4\sqrt{3} - 6} = -\frac{1}{\sqrt{3}}$$

so 2θ is 150° (restricting to the range $0 \leq \theta \leq 180^\circ$). So θ must be 75° .

- Differentiate to get $1 - x^{-2}$ which is zero at $x = \pm 1$. The minimum value of the function occurs when $x = 1$, and the value is 2.

Alternatively, use the fact that $a^2 + b^2 \geq 2ab$ with $a = x^{1/2}$ and $b = x^{-1/2}$. Then $x + x^{-1} \geq 2x^{1/2}x^{-1/2} = 2$.

MAT 2010 Q4

- (i) If I drop a perpendicular from $(1, 2h)$ to $(1, 0)$ then I have a right-angled triangle with angle θ , opposite a side of length $2h$ and adjacent to a side of length $2h$. So $\tan \theta = 2h$.
- (ii) $(1, 2h)$ lies in $x^2 + y^2 < 4$ if and only if $1 + 4h^2 < 4$, which rearranges to $h^2 < \frac{3}{4}$. Since $h > 0$, this condition is equivalent to $h < \sqrt{3}/2$.
- (iii) The gradient of the line is $-h$ because the y -value changes by $-2h$ between $x = 1$ and $x = 3$. The line goes through $(3, 0)$ and has equation $y = -h(x - 3)$.

We could look for repeated roots between $x^2 + y^2 = 4$ and $y = -h(x - 3)$. In general if I substitute one into the other I get $x^2 + h^2(x - 3)^2 = 4$. If $h = 2/\sqrt{5}$ then this is $5x^2 + 4(x - 3)^2 = 20$. If we multiply this out, rearrange, and factorise, we get $(3x - 4)^2 = 0$ indicating that there is a double root at $x = 4/3$, so the line is tangent to the circle.

- (iv) In this case, the point $(1, 2h)$ is outside the circle, because $\frac{2}{\sqrt{5}} > \frac{\sqrt{3}}{2}$ (I know this because $\frac{4}{5} > \frac{3}{4}$). The diagram is like the first picture in the question. The area inside both is the area of the sector with angle θ . So the area is $4\pi \frac{\theta}{360^\circ}$ where $\tan \theta = 2h$.

- (v) In this case, the point $(1, 2h)$ is inside the circle, because $\frac{6}{7} < \frac{\sqrt{3}}{2}$ (I know this because $\frac{36}{49} < \frac{3}{4}$ (I know this because $144 < 147$)). The diagram is like the second picture in the question.

Check that $(8/5, 6/5)$ lies on the circle; $(8/5)^2 + (6/5)^2 = (64/25) + (36/25) = 4$. Check that $(8/5, 6/5)$ lies on the line; $-\frac{6}{7}(\frac{8}{5} - 3) = \frac{6}{5}$.

The area is made up of a triangle with corners $(0, 0)$ and $(1, 12/7)$ and $(8/5, 6/5)$, plus a sector of a circle from the x -axis up to the line from the origin to $(8/5, 6/5)$.

Using the formula in the question, the area of the triangle is $\frac{27}{35}$. The area of the sector is $4\pi \frac{\alpha}{360^\circ}$ where $\tan \alpha = \frac{6}{8}$. Add these.

Extension

- Let A be (a, b) and let C be (c, d) and let M be $(a, 0)$ and let N be $(c, 0)$. The area we want is the triangle OCN plus the trapezium $NCAM$ minus the triangle OAM . If we find all of these and simplify, we get $\frac{1}{2}(ad - bc)$. The absolute value signs come in because I haven't been very careful; I've assumed that (a, b) and (c, d) are a particular way around.
- Consider the cross product of the vectors $(a, b, 0) \times (c, d, 0) = (0, 0, ad - bc)$. We also know that the magnitude of $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}|\sin \theta$, which is (almost) exactly the formula for the area of the triangle. So we just need to take the magnitude of $\mathbf{a} \times \mathbf{b}$ and divide by 2 to get the area of the triangle.