DSWs and Traveling Wave solutions of Fifth Order KdV Equations

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INI Visit to Oxford
22 September 2022

Acknowledgements

Collaborators

Michael Shearer
Mark Hoefer
Tom Bridges

Funding

NSF-DMS grant 812445
INI-Simons fellowship

References

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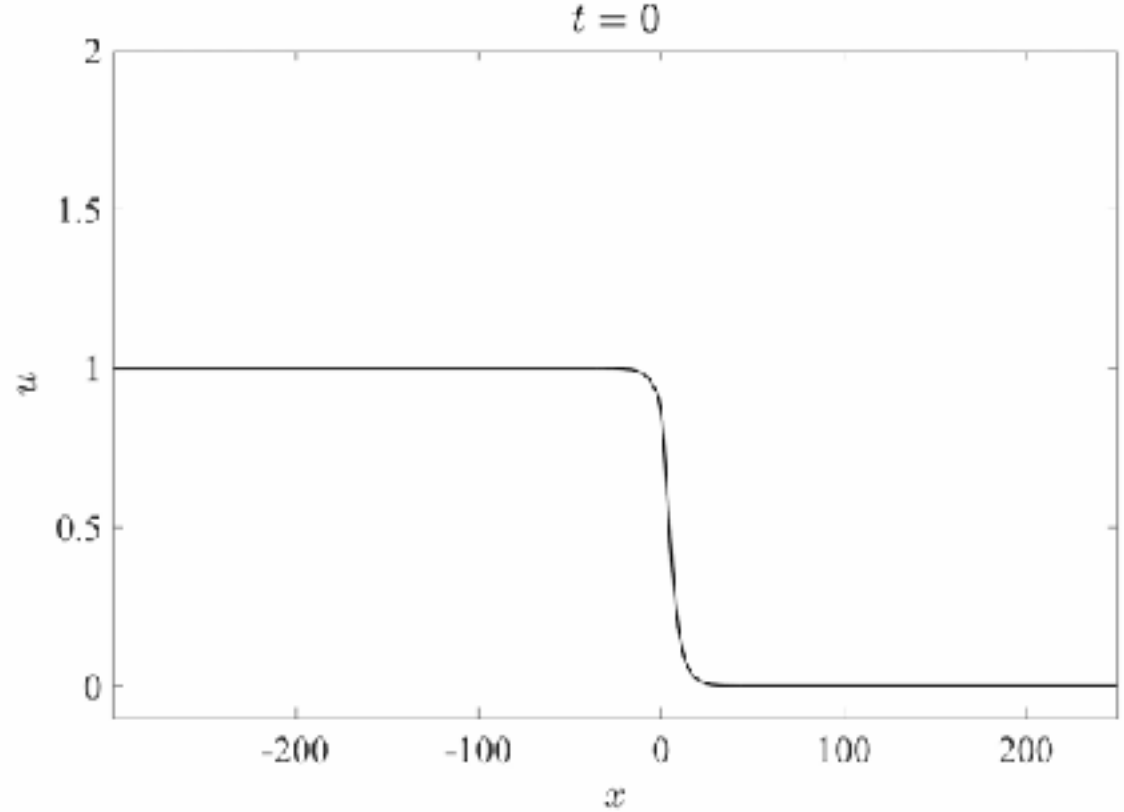
Korteweg-de Vries Equation: DSWs

$$u_t + uu_x + u_{xxx} = 0$$

$$u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

(smoothed)



Gurevich-Pitaevskii solution:
Modulated elliptic function

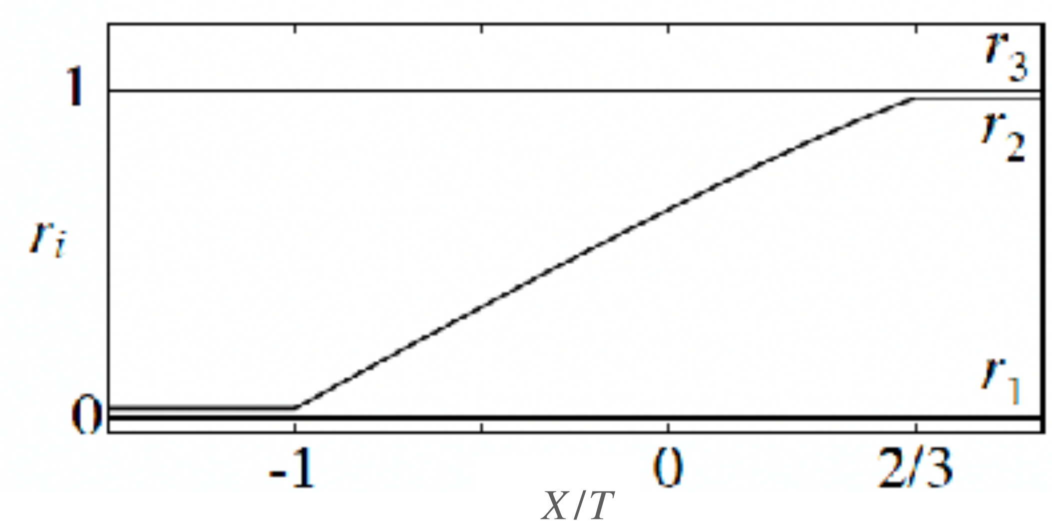
[Gurevich and Pitaevskii JETP (1974)]

$$u(\theta) = r_1 + r_2 - r_3 + 2(r_3 - r_1)\mathrm{dn}^2\left(\sqrt{\frac{r_1 - r_3}{6}}\theta; m\right) \qquad \theta_X = k,\ \theta_T = -\omega$$

$X = \epsilon x,\ T = \epsilon t$

$$\mathbf{r}_T + V_i(\mathbf{r})\mathbf{r}_X = 0 \quad \mathbf{r} = [r_1\ r_2\ r_3]^{\mathrm{T}}$$

DSW corresponds to rarefaction wave solution of diagonalized Whitham modulation equations



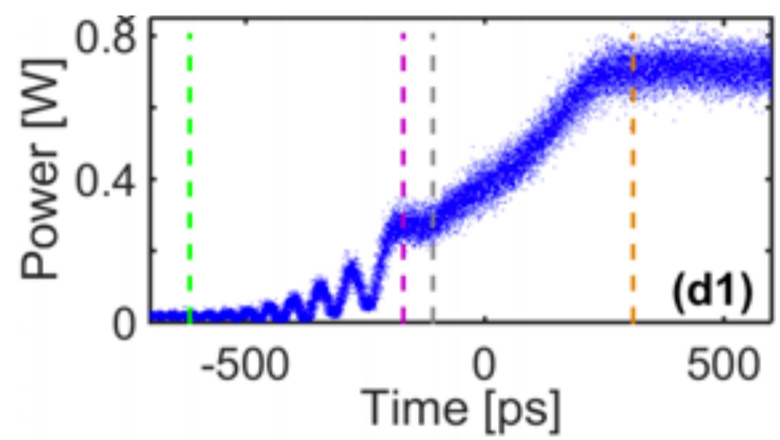
DSWs in dispersive hydrodynamics

Internal/surface water waves

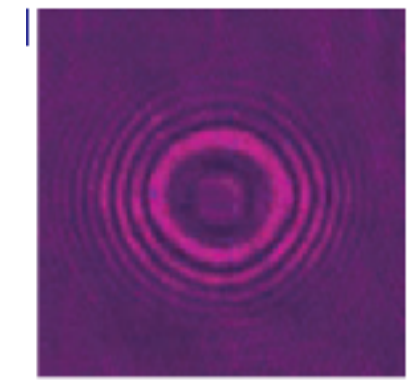
NASA Terra MODIS

S. Dickerson/Red Bull Illume

Nonlinear optics

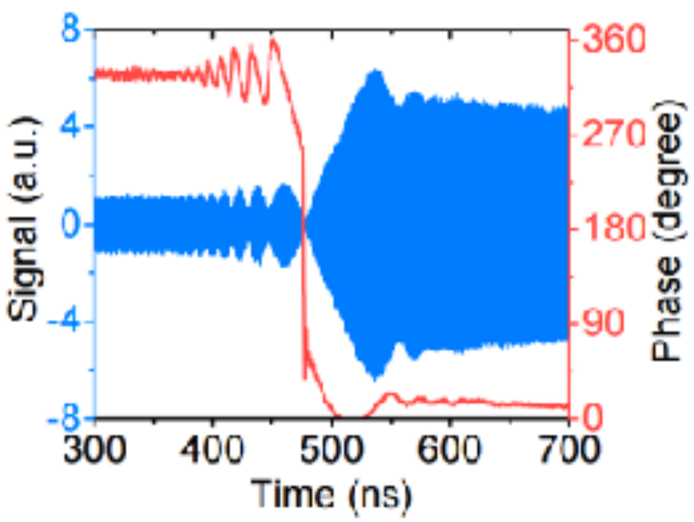


Xu et al. PRL**118**, 254101 (2017)



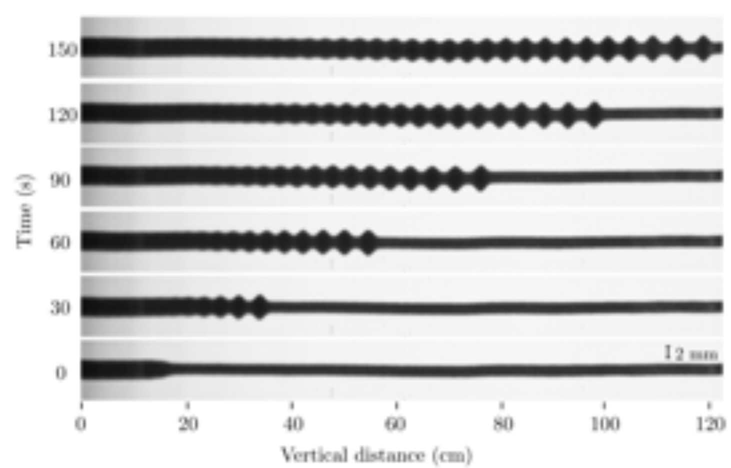
WAN et al. Nat. Phys. **3** (2007)

Spin waves magnetic films



Janantha, et al. PRL **119**, 024101 (2017)

Viscous fluid conduits



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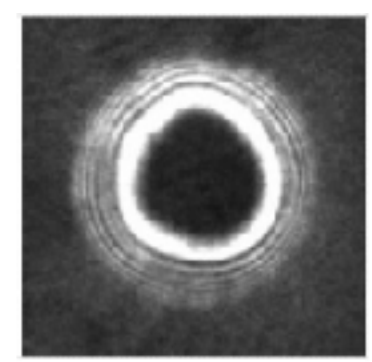
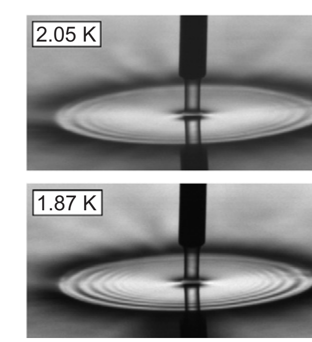
Superfluids



Hoefer et al. PRA, **74,** 023623 (2006)



Rolley et al. Physica B **394** (2007)

Kawahara Equation

Model for weakly nonlinear long waves in the presence of strong surface tension [Hunter and Scheurle Physica D (1998)]

$$\eta_t + \frac{3c_0}{2h}\eta\eta_x + \frac{1}{2}c_0h^2(\tfrac{1}{3} - B)\eta_{xxx} + \frac{c_0h^4}{90}\eta_{xxxxx} = 0$$

Rescaled equation $u_t + uu_x + \sigma u_{xxx} + u_{xxxxx} = 0, \qquad \sigma = \pm 1, 0$

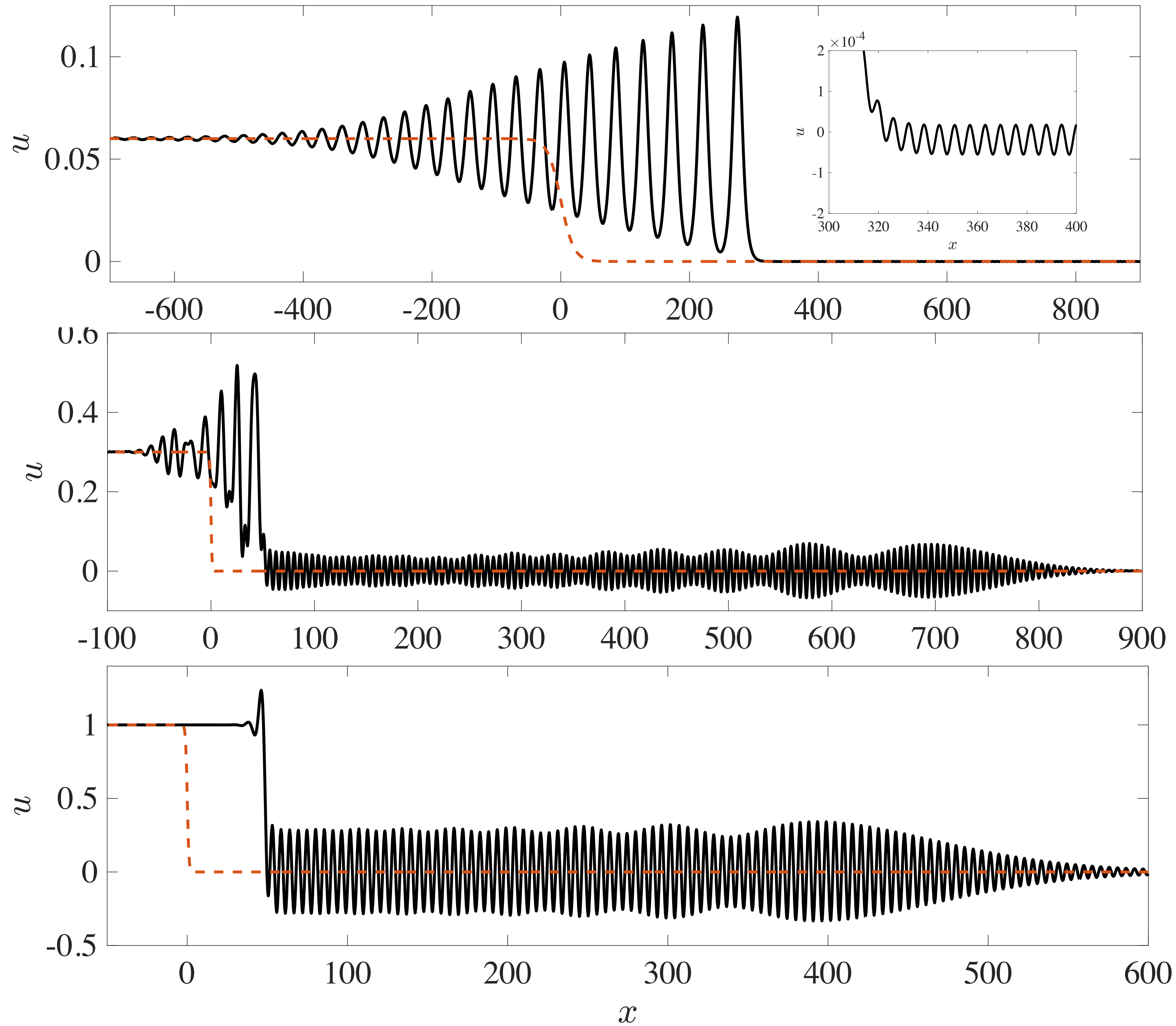
Dispersion relation $\omega(k;\bar{u}) = \bar{u}k - \sigma k^3 + k^5$

Applications

- Flexural ice sheets [Marchenko, PMM USSR **52**(2) 1988]
- Nonlocal, nematic liquid crystals [Smyth, El Proc Roy Soc A, 472 (2016)]
- Spin-orbit coupled BEC [Khamehchi PRL 118, 155301 (2017)]
- Chains of electromagnetic oscillators [Gorshov et al. Phys Lett. 74 (1979)]
- Collisionless plasma [Kakutani, Ono JPS 26 (1969)]

Kawahara Riemann problem

$$u_t + uu_x + \sigma u_{xxx} + u_{xxxxx} = 0, \quad \sigma = \pm 1 \qquad u(x,0) = \begin{cases} \Delta & x < 0 \\ 0 & x > 0 \end{cases}$$



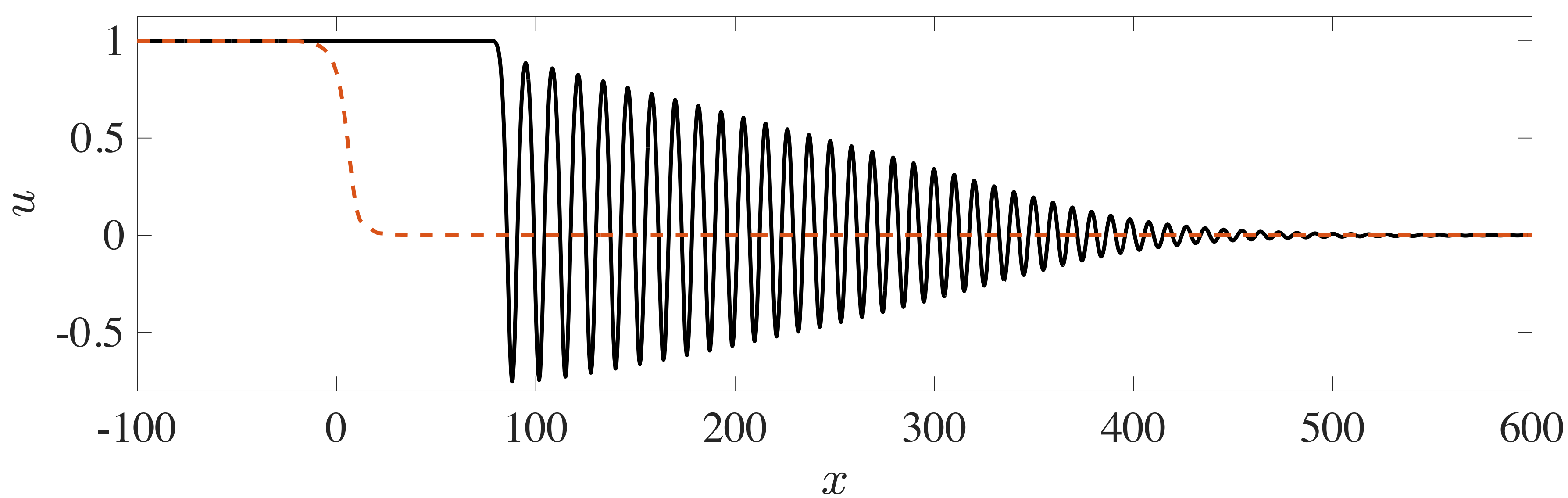
Case: $\sigma = +1$

Increasing Δ

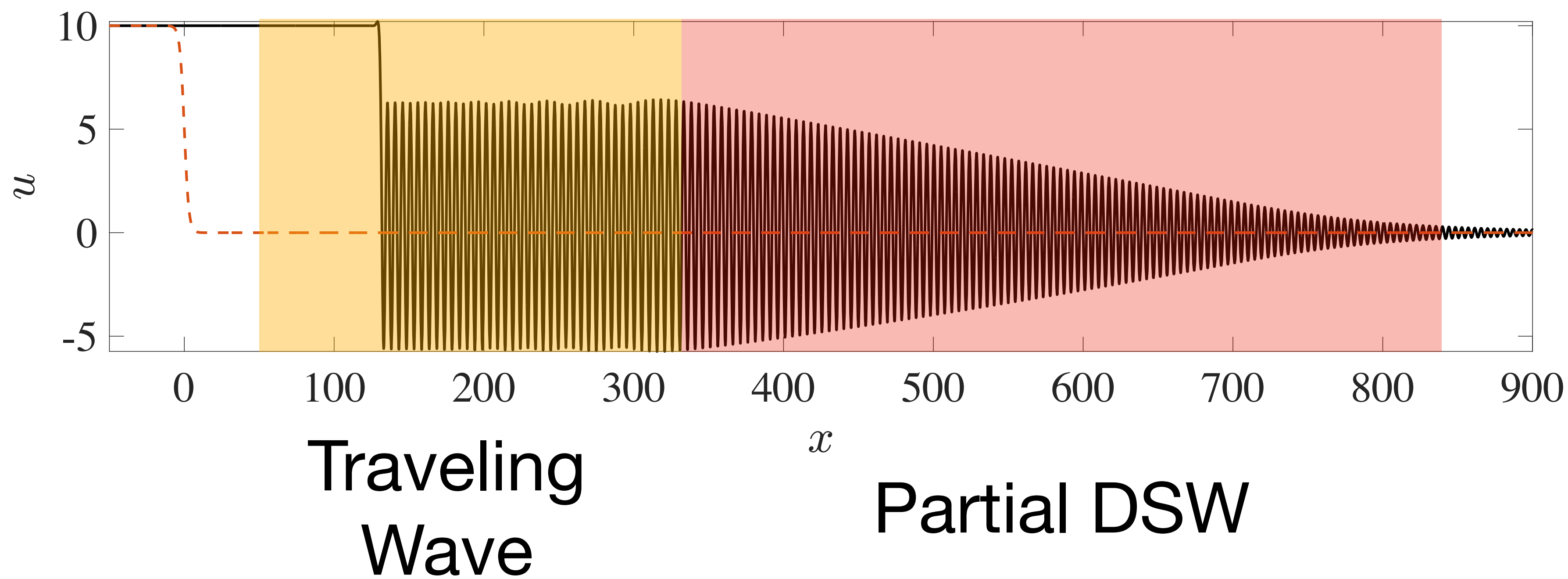
Detailed study of DSW structure found in PS, Hoefer SIAM J Appl Math 2017

Kawahara Riemann problem

$$u_t + uu_x + \sigma u_{xxx} + u_{xxxxx} = 0, \quad \sigma = \pm 1 \qquad u(x,0) = \begin{cases} \Delta & x < 0 \\ 0 & x > 0 \end{cases}$$



Case: $\sigma = -1$



For sufficiently large amplitude jumps, the resulting DSW resembles those for $\sigma = 0$

[Hoefer, Smyth, PS Stud. Appl Math 142 (2018)]

Construction of TW solutions

Seek traveling wave solution of form $u = u(x - ct)$

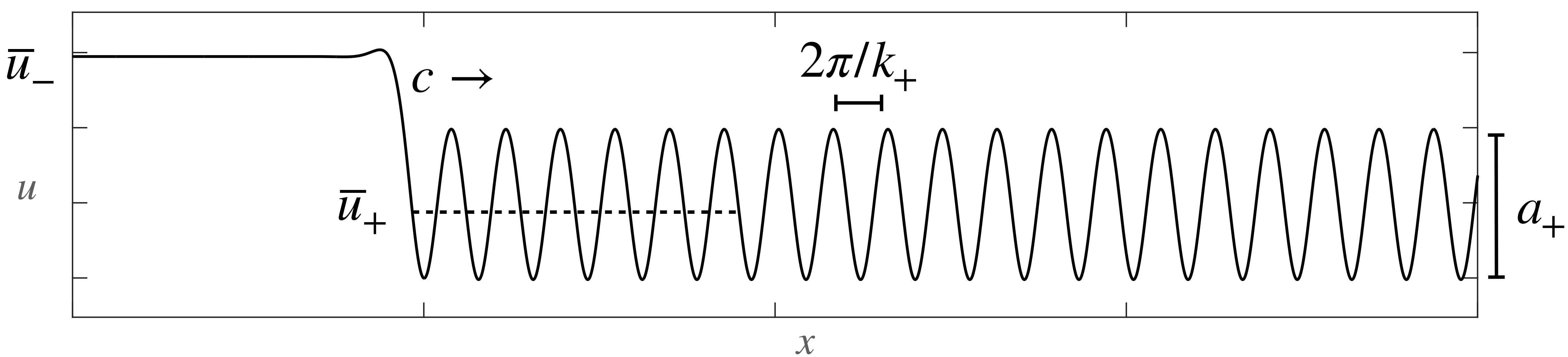
$$-cu + \frac{1}{2}u^2 + \sigma u'' + u^{(4)} = \mathscr{A}$$

Corresponding traveling wave Hamiltonian

$$\mathscr{H} = -\frac{c}{2}u^2 + \frac{1}{6}u^3 + \frac{\sigma}{2}(u')^2 + u'''u' - \frac{1}{2}(u'')^2 - \mathscr{A}u,$$

Traveling wave jump conditions: $[\![\mathscr{A}]\!] = 0$ $\quad c = c_{\mathrm{p}}$

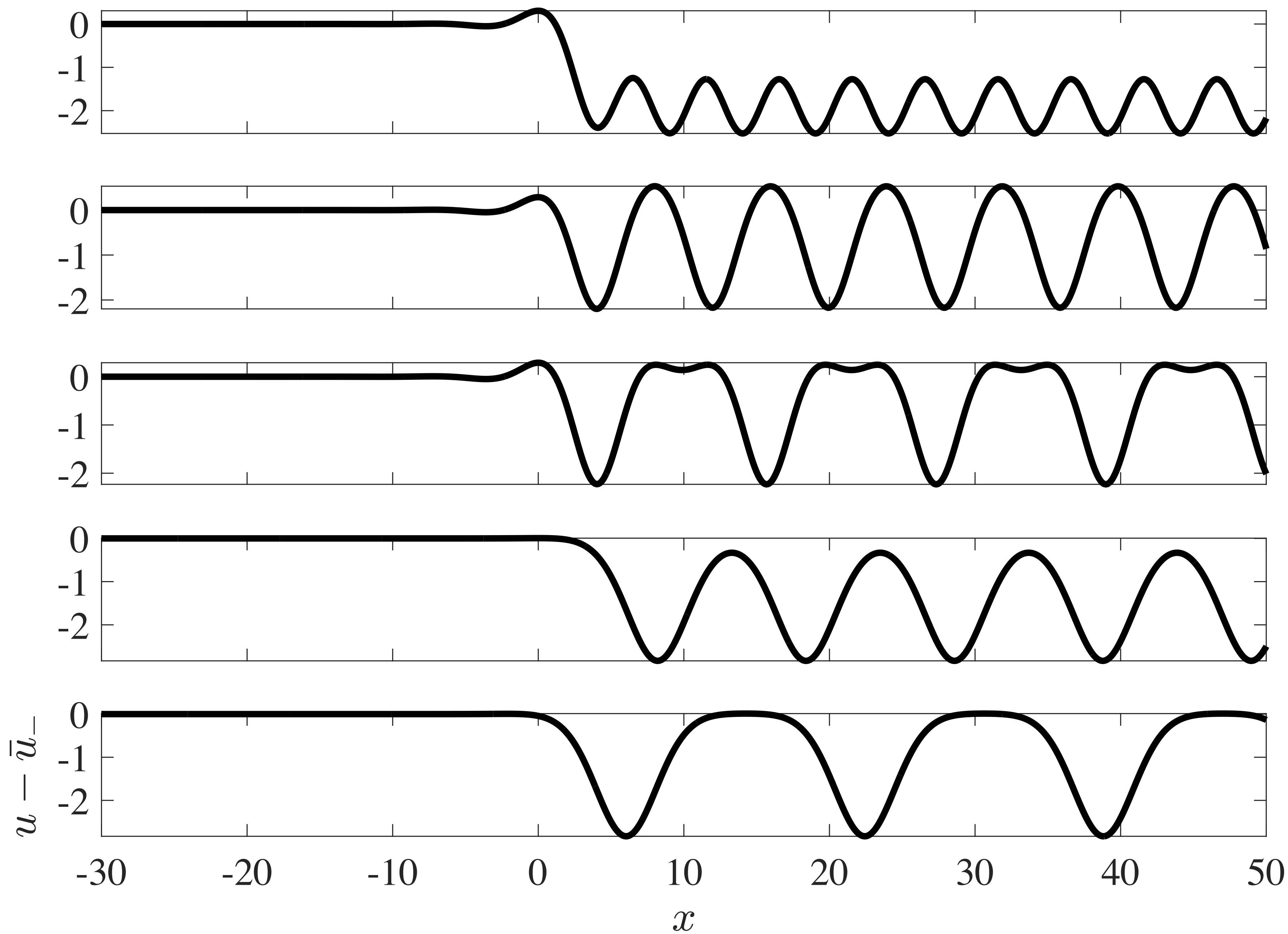
$$[\![\mathscr{H}]\!] = 0$$



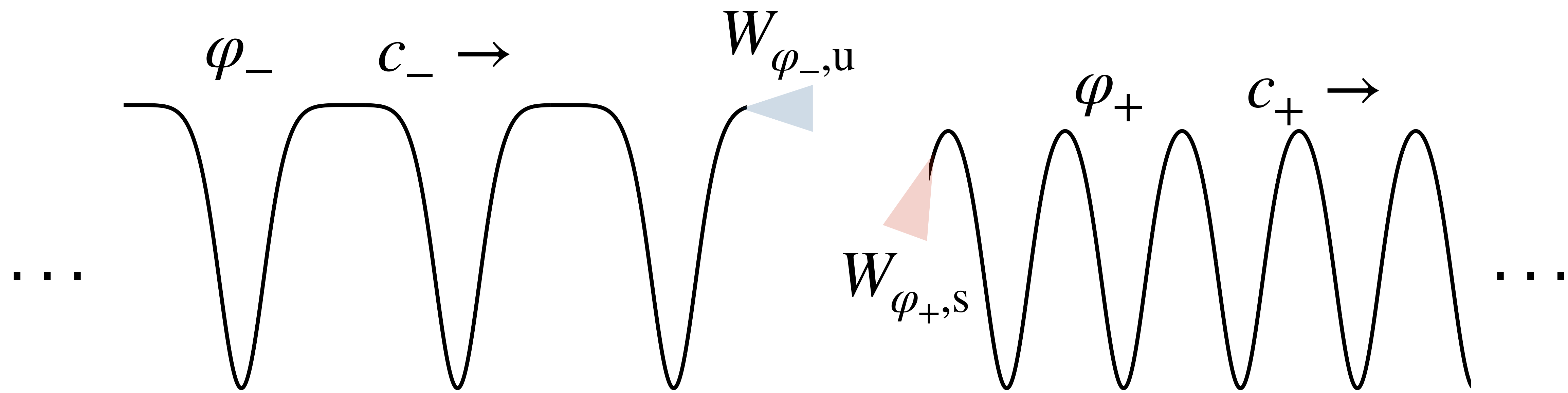
Equilibrium-to-periodic TWs

Solutions computed on periodic domain using Newton-CG method [Yang 2009]

Numerical computations are similar to those for “multi-pulsed” solitary waves [Buffoni, Champneys, Toland J. Dyn. Diff. Eq (1994)]



Periodic-to-Periodic



Assume the existence of two distinct periodic orbits with the same velocity

Constant Hamiltonian $\Longrightarrow$ three dimensional subspace of 4D phase space

Computation two dimensional invariant manifolds of each hyperbolic periodic orbit and their on appropriately chosen Poincare section yields heteroclitic orbit between far-field states

Connection to Modulation theory

Nonlinear modulated wave

$$u = \varphi(\theta; \bar{u}, a, k)$$

$$\theta_X = k \quad \theta_T = -\omega$$

$$(\overline{u})_T + \frac{1}{2}\left(\overline{\varphi^2}\right)_X = 0$$

$$\left(\overline{\varphi^2}\right)_T + \left(\frac{2}{3}\overline{\varphi^3} - 3k^2\sigma\overline{\varphi_\theta^2} + 5k^4\overline{\varphi_{\theta\theta}^2}\right)_X = 0$$

$$k_T + \omega_X = 0$$

Whitham shocks satisfy jump conditions

$$-c(\overline{u}_- - \overline{u}_+) + \frac{1}{2}\left(\overline{\varphi_-^2} - \overline{\varphi_+^2}\right) = 0$$

$$-\frac{c}{2}\left(\overline{\varphi_-^2} - \overline{\varphi_+^2}\right) + \frac{1}{3}\left(\overline{\varphi_-^3} - \overline{\varphi_+^3}\right) + \frac{3}{2}\sigma\left(k_-^2\overline{\varphi_{-,\theta}^2} - k_+^2\overline{\varphi_{+,\theta}^2}\right) - \frac{5}{2}\left(k_-^4\overline{\varphi_{-,\theta\theta}^2} - k_+^4\overline{\varphi_{+,\theta\theta}^2}\right) = 0$$

$$-c(k_- - k_+) + \left(c_-k_- - c_+k_+\right) = 0$$

Far-field parameters of periodic orbits correspond to solutions of the generalized Riemann problem

$$(\bar{u}, a, k) = \begin{cases} (\bar{u}_-, a_-, k_-) & x < 0 \\ (0, a_+, k_+) & x > 0 \end{cases}$$

Whitham shocks

Theorem: Traveling waves connecting two distinct periodic waves satisfy jump conditions of Whitham Eqs.

$$-c[\![\mathbf{P}]\!] + [\![\mathbf{Q}]\!] = 0$$

Definition: A Whitham shock is **admissible** if ∃ traveling wave with far-field periodic wave behavior given by jump conditions.

Numerical computations suggest that stable Whitham shocks are *undercompressive*

Analysis of jump conditions:
Bifurcation from periodic waves to two distinct far-field periodic orbits

Computation of traveling waves:
Find intersections of unstable manifolds of periodic orbits

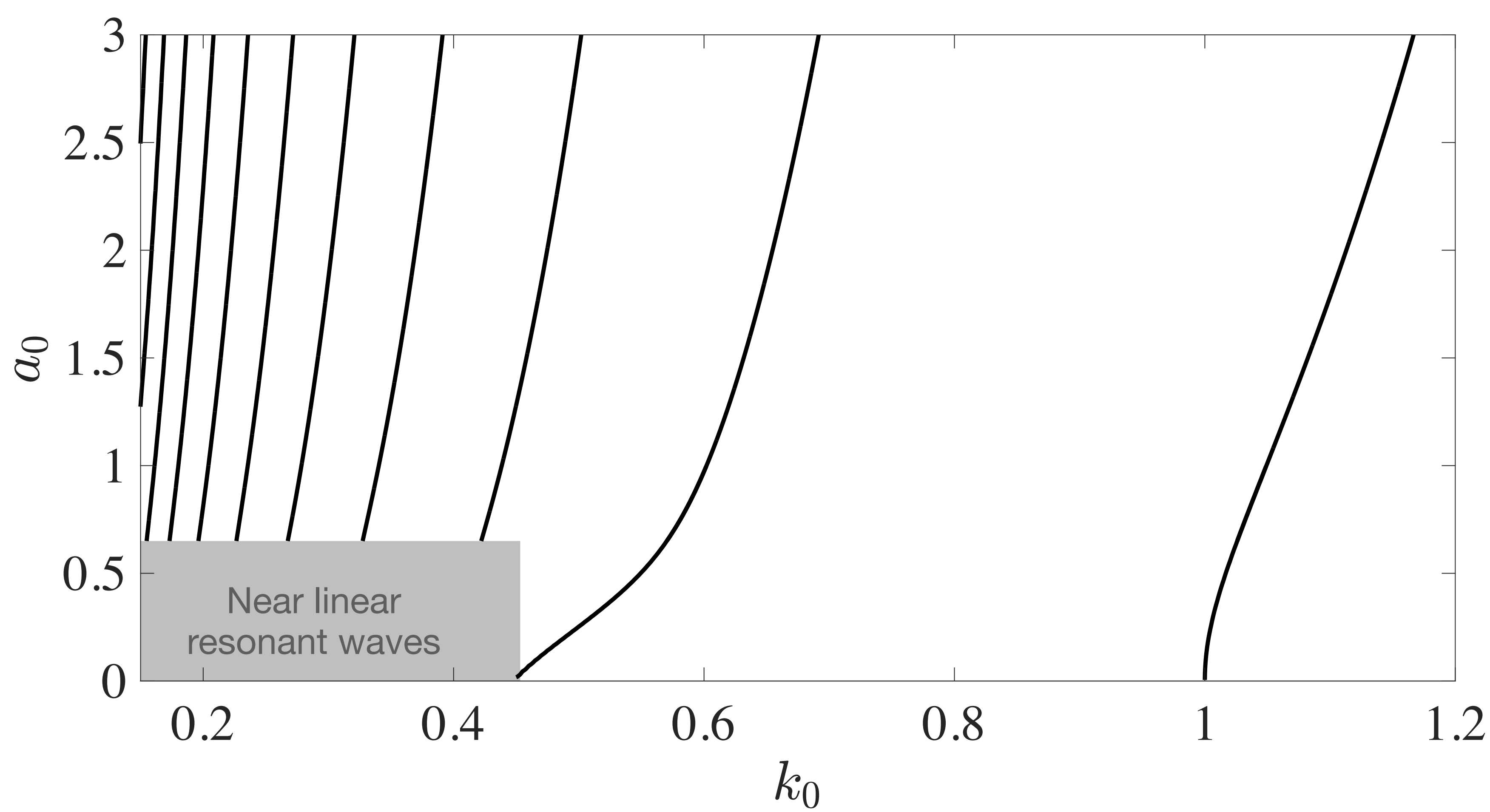
Bifurcations from periodic waves

Linearize jump conditions around trivial solution $\mathbf{F}(\mathbf{q_0}) = \mathbf{0}$

$$\mathbf{q_0} = (0, a_0, k_0)$$

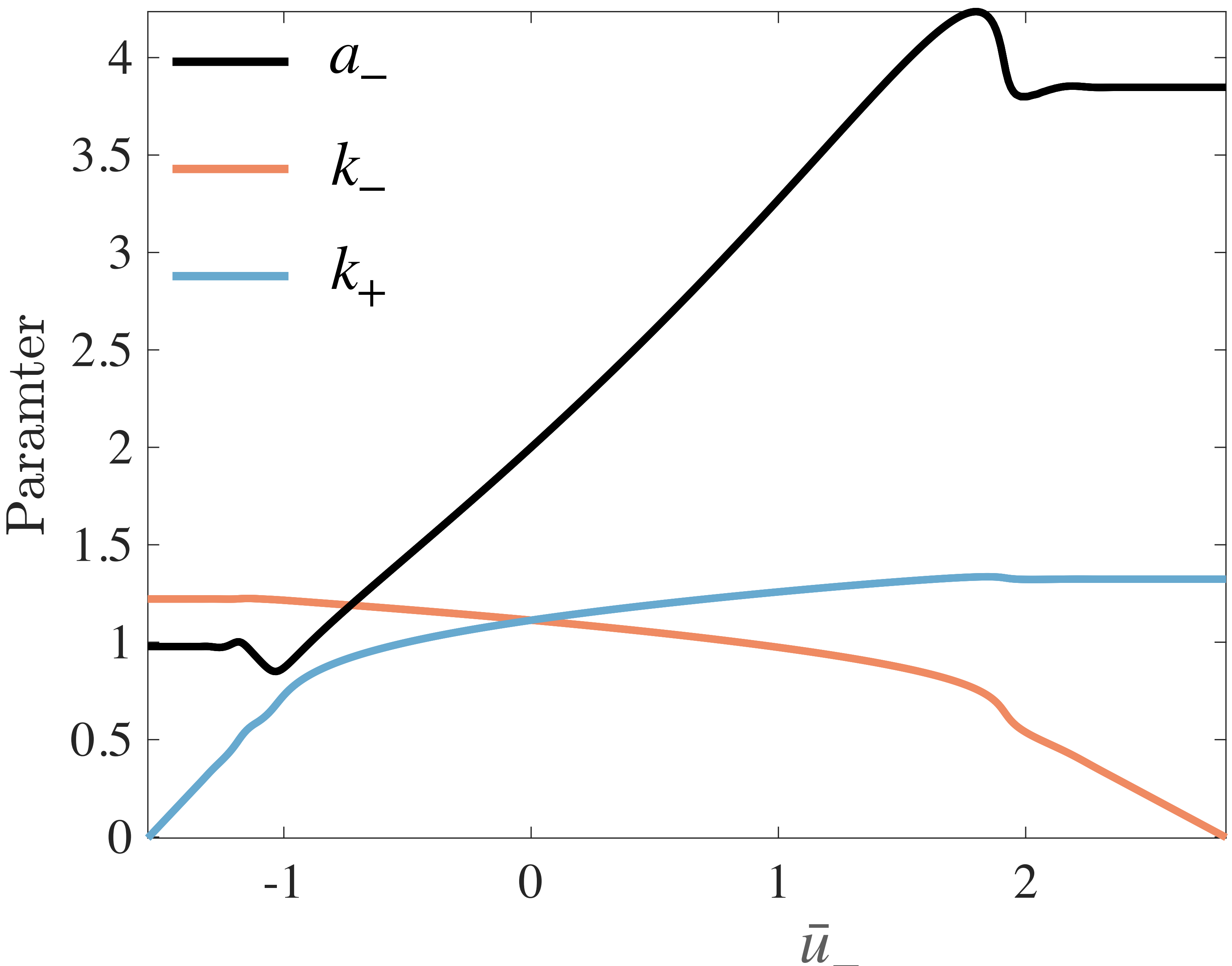
Compute parameters for which $\left| \nabla_{\mathbf{q}} F(\mathbf{q}_0) \right| = 0$

Example: Numerical computations with $\sigma = +1$



Traveling wave loci

Families of solutions of jump conditions yields 5 far-field wave parameters: $(u_-, a_-, k_-, a_+, k_+)$



Example of computed solution locus for fixed $a_+ = 2$

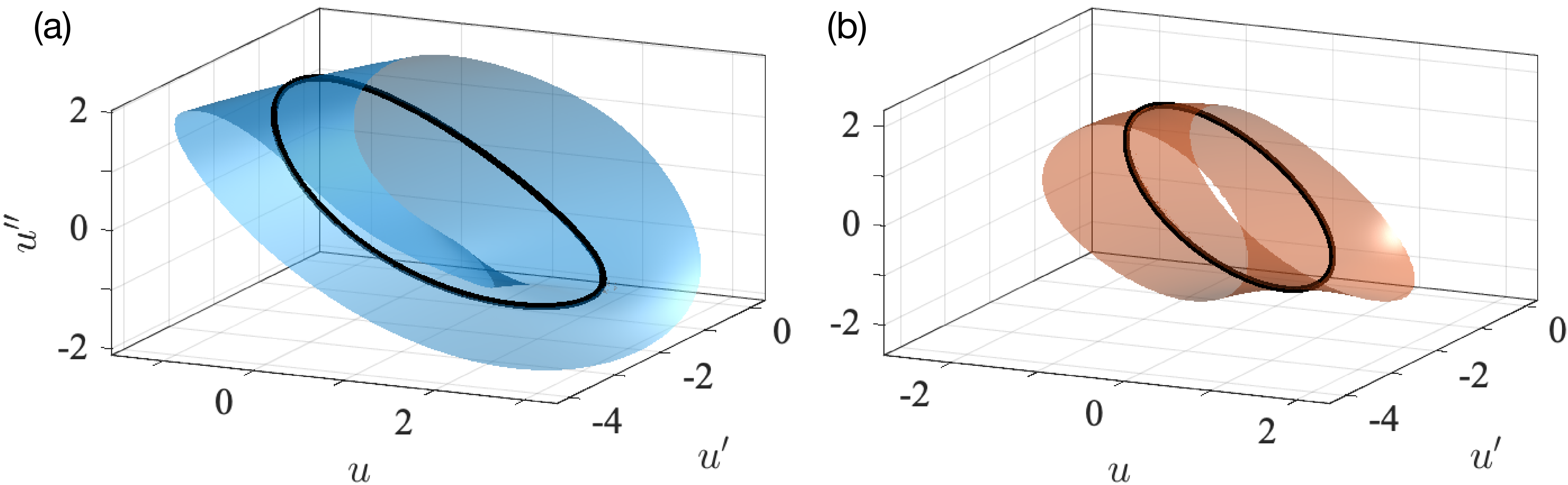
Limiting cases: $k_\pm \to 0$

Procedure to construct TW:

- Compute Floquet multipliers
- Compute corresponding stable/unstable manifolds
- Find intersection on appropriate Poincaré section

Constructing TWs

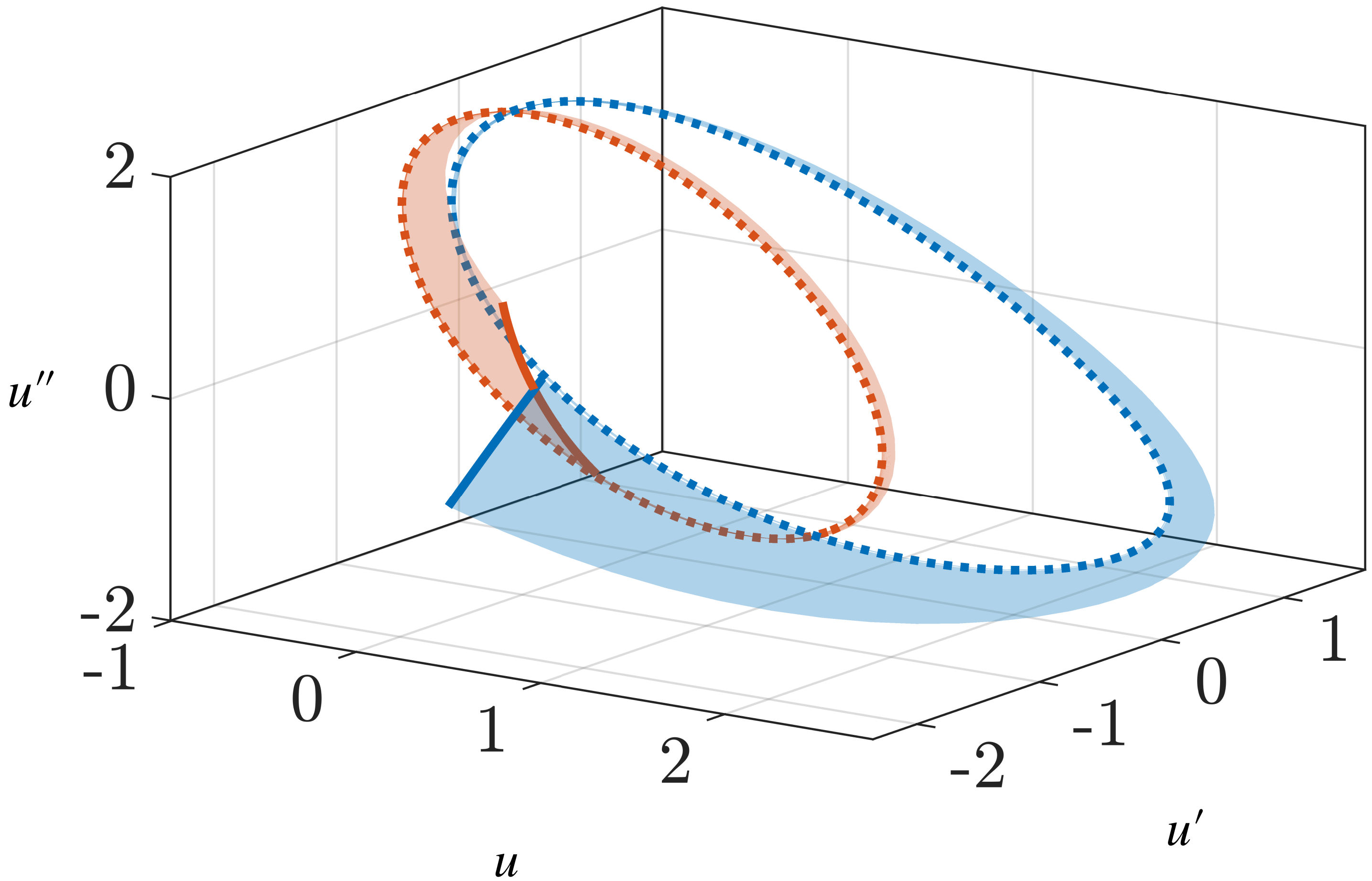
Suppose $\varphi_{\pm}$ are periodic orbits with real Floquet multipliers



(a) 2D unstable manifold of φ_-: Möbius strip, $\lambda < -1$

(b) 2D stable manifold of φ_+: cylinder, $0 < \lambda < 1$

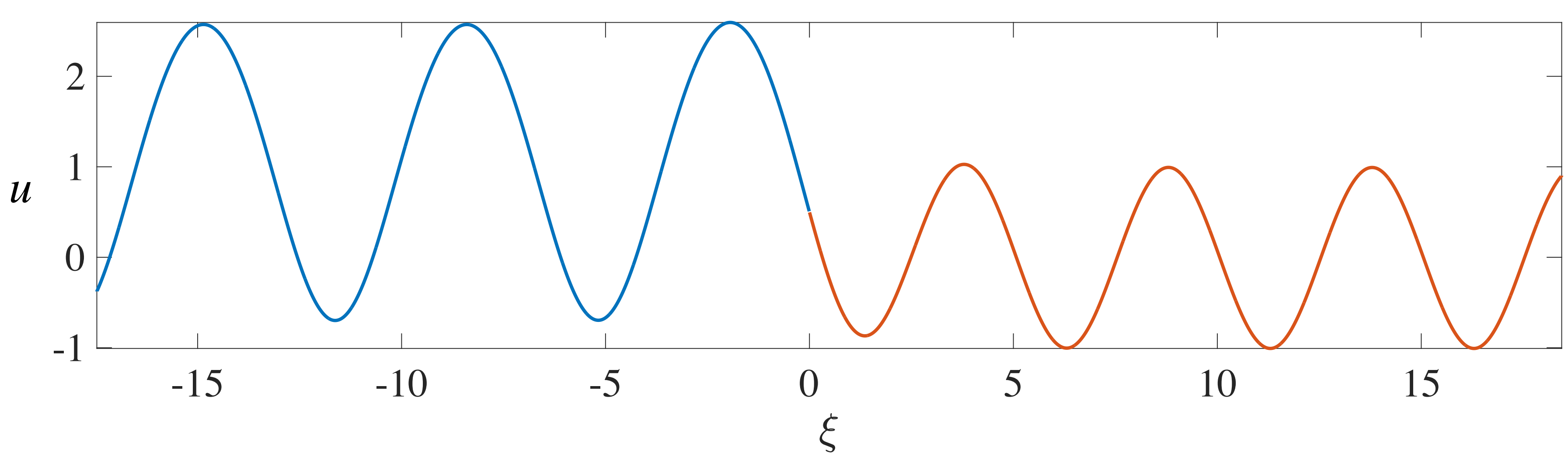
Computations of traveling waves



Computations of invariant manifolds of two hyperbolic periodic orbits φ_- and φ_+

φ_-

φ_+



Traveling wave reconstructed from intersections of invariant manifolds at $\xi = 0$

Extension to systems

Boussinesq systems in shallow water hydrodynamics

[Bona, Chen & Saut J. Nonlinear Sci (2002)]

$$a, b, c \in \mathbb{R}$$

$$\begin{aligned} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} &= 0 \\ u_t + \eta_x + u u_x + c\eta_{xxx} - b u_{xxt} &= 0 \end{aligned}$$

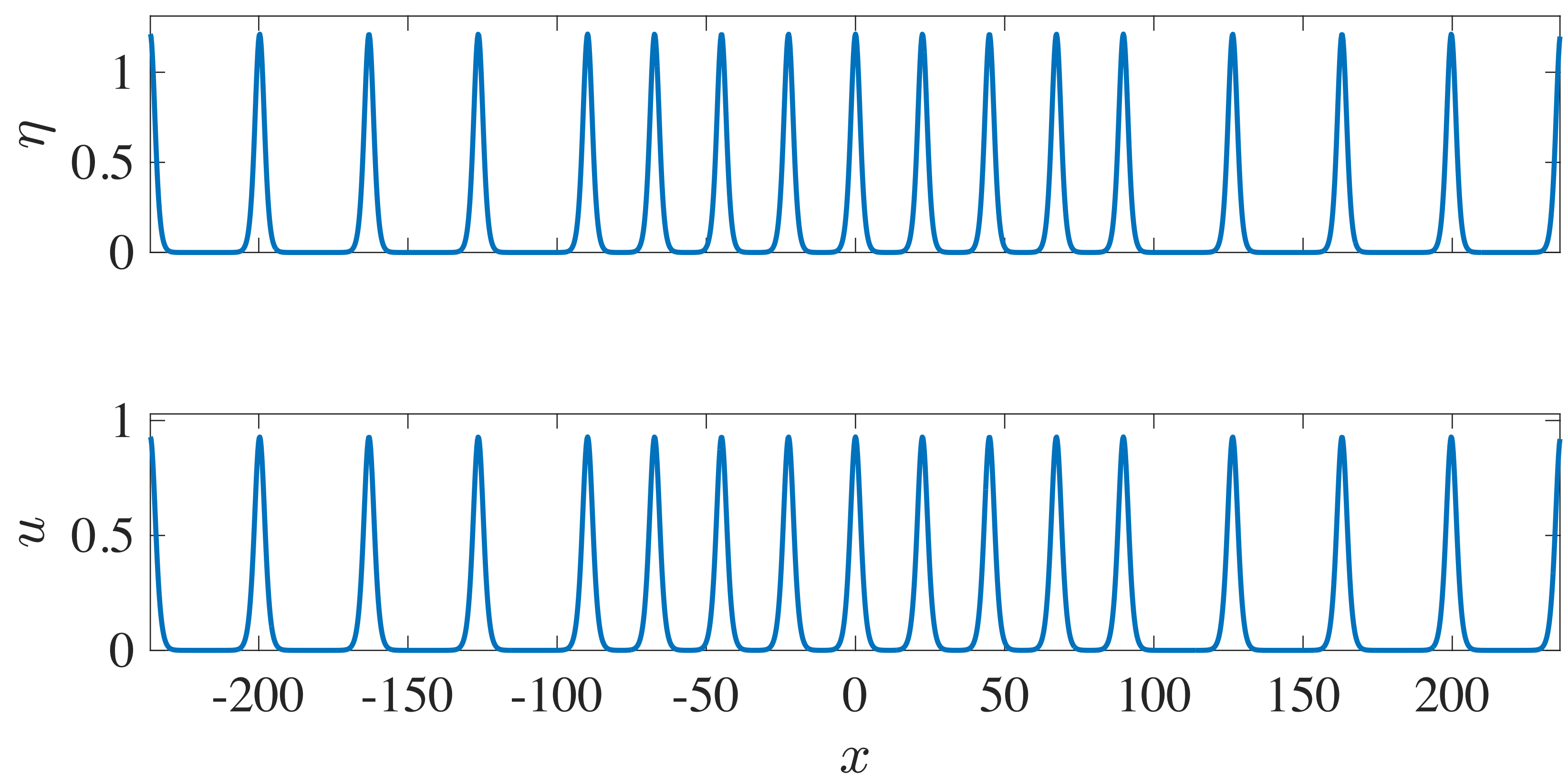
$$a + 2b + c = \frac{1}{3}$$

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)$$

$$b + c = \frac{1}{2}\left(1 - \theta^2\right)$$

Traveling wave equations form Hamiltonian dynamical system in $\mathbb{R}^4$

Preliminary numerical computations suggest existence of localized, oscillatory defects on periodic background



Conclusions

Presence of fifth order dispersion results in nonclassical DSW structure

Portion of nonclassical DSWs correspond to a Whitham shock that satisfy RH conditions for Whitham modulation equations

Traveling waves bifurcate from degenerate periodic orbits that are at the hyperbolic-elliptic transition

Computations of periodic-periodic traveling waves are successful so long as far-tied periodic orbits are both hyperbolic

Open avenues for future work

- Extensions to systems in shallow water hydrodynamics
- Extensions to nonlocal equations, e.g. Ostovsky, Whitham etc

Thank you!