Existence of hypersurfaces with prescribed mean curvature (PMC)

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(joint work with Neshan Wickramasekera, Cambridge)

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Natural existence question: prescribing mean cuvature

$(N^{n+1}, \mathtt{h})$ compact Riem mfld; $g : N \to \mathbb{R}$ sufficiently regular.

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Assume $g \geq 0$ and Lipschitz ($C^{0,1}$), $n \geq 2$. There exists $M = M^n$ C^2-immersed, two-sided ($\exists$ unit normal ν), the mean curvature of M is $g\nu$, and $\dim(\overline{M} \setminus M) \leq n - 7$ (small "singular set"). More precisely, M is a quasi-embedding.

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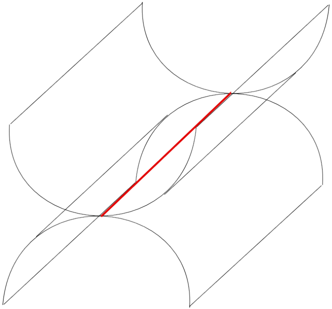
Quasi-embedding: immersion that fails to be an embedding only due to tangential self-intersections;
around any non-embedded point "two embedded disks, lying on one side of each other, intersecting tangentially".

Natural existence question: prescribing mean cuvature



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Singular set expected in high dimensions (it is characteristic already of area-minimisers).

Mean curvature $\vec{H}$ related to n-area properties:

$\vec{H}$ is the *"negative gradient"* of n-area, i.e.

deform n-dim. hypersurface with speed $\vec{H}$ to (instantaneously) decrease n-area fastest.

Notes:

- M is $C^{2,\alpha}$-immersed for every $\alpha \in (0,1)$. If g is $C^{k,\alpha}$ for $k \geq 1$, then M is $C^{k+2,\alpha}$.
- No singular set ($M = \overline{M}$) for $n \leq 6$.
- The theorem applies for $g \leq 0$ (switch the normal).
- $g \equiv \lambda \in \mathbb{R}$: CMC (constant-mean-curvature) hypersurface (minimal for $g \equiv 0$).

Which functional to use

Seek PMC hypersurface with mean curvature g $\rightsquigarrow$ functional

$$J_g(\Omega) = \underbrace{\text{Perimeter}(\Omega)}_{\mathcal{H}^n(\partial\Omega)} - \int_\Omega g \, d\mathcal{H}^{n+1} \text{ (or something similar?)}$$

and look for critical points. (For $g \equiv 0$ it's the area functional.)

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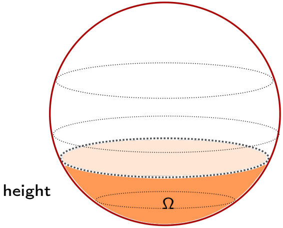

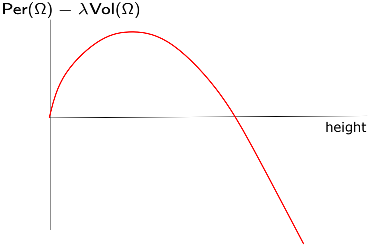

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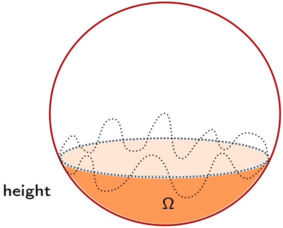
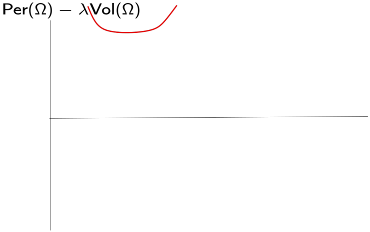
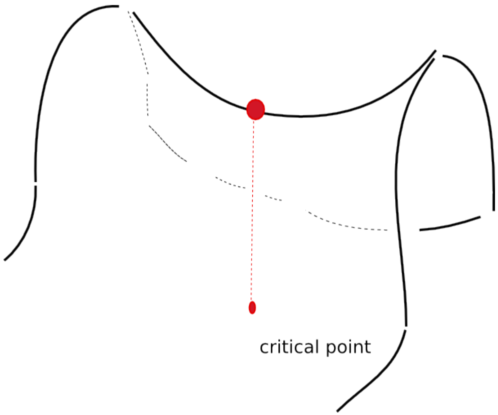




Finding critical points of “saddle type”: minmax methods



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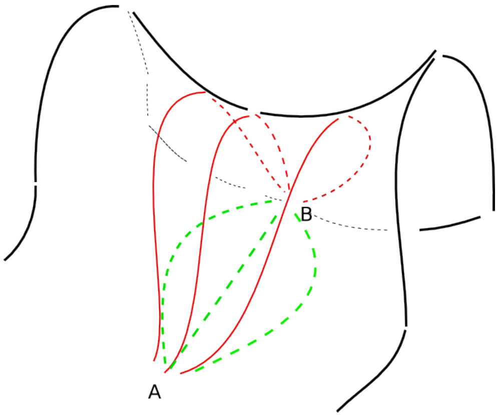

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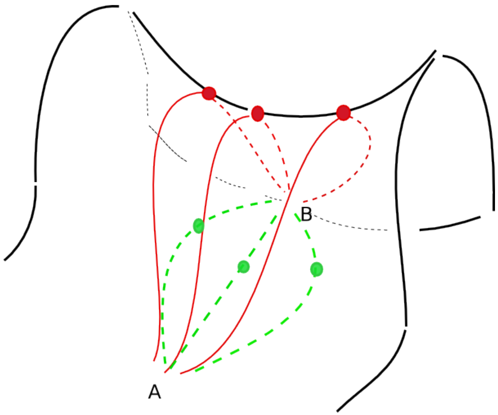

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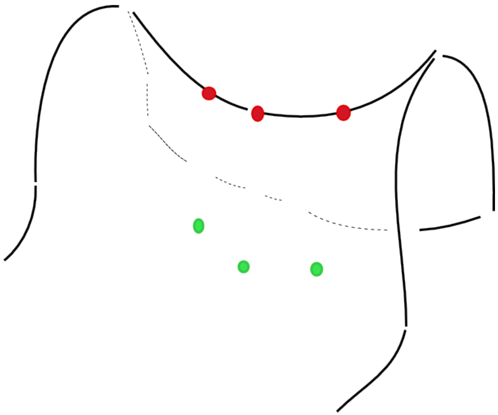

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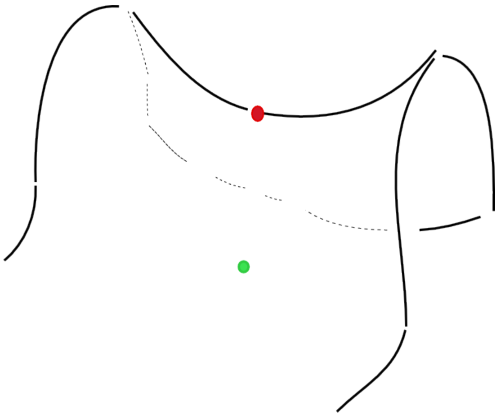

Finding critical points of "saddle type": minmax methods

Can we extract limit to produce critical point?

Classical (70s) PDE mountain pass (minmax) methods: functional defined on a Hilbert space, with validity of 'Palais–Smale condition' (suitable compactness property).

Limits of smooth submanifolds: singularities and multiplicity

Embed smooth hypersurfaces in the space of *varifolds*. (Almgren)

Vari~~ational mani~~fold: very weak GMT notion of submanifold, with weak topology, makes it "easier" to extract limits.

But... with a potentially huge singular set and with multiplicity!

Unless wide singular behaviour is ruled out, the object is not something that anyone would call a hypersurface.

In other words: will analysis find the object that geometry wants?

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What we need on the GMT side

Expectation/hope from minmax construction: Morse index ≤ 1.

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[Bel.–Wickramasekera '18, '19]:
C^2-regularity and compactness for *a class of* integral n-varifolds under stationarity and finite Morse index conditions w.r.t J_g.

PMC existence minmax methods

To find g-PMC hypersurface: minmax using functional akin to

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[Dey, '19]: existence of CMC hypersurfaces; extends [Zhou-Zhu '19] to $n > 6$ employing [Bel.-Wickramasekera '18, '19: regularity and compactness theory].

Minimal case

Existence of a closed minimal hypersurface (originally Almgren '66, Pitts '77, Schoen–Simon '80 via Almgren–Pitts method).

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Via *Allen–Cahn* mountain pass (minmax):
[Guaraco '18], [Hutchinson–Tonegawa '00], [Tonegawa '05], [Wickramasekera '14], [Tonegawa–Wickramasekera '12].

Do not work directly with area functional. Instead:
for $\varepsilon > 0$ work with regularised energy $\mathcal{E}_\varepsilon$, defined on $W^{1,2}(N^{n+1})$, *with* Palais–Smale condition. Euler– Lagrange eqn elliptic semi-linear. *Classical PDE minmax on Hilbert space.*
$\mathcal{E}_\varepsilon \overset{\varepsilon\to 0}{\to}$ n-area (in a strong sense).
Recover minimal hypersurface in the $\varepsilon \to 0$ limit.

Back to PMC existence, via Allen–Cahn minmax

Today's talk: *Allen–Cahn* minmax method for the main PMC existence theorem (first slide).

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Do not work with J_g directly.
For $\varepsilon > 0$ work with regularised energy $\mathcal{F}_{\varepsilon,g}$, defined on $W^{1,2}(N^{n+1})$, with Palais–Smale condition. Euler– Lagrange eqn semi-linear. Classical PDE minmax on Hilbert space.

Hopes: $\mathcal{F}_{\varepsilon,g} \overset{\varepsilon\to 0}{\to} J_g$; find the desired PMC hypersurface in the $\varepsilon \to 0$ limit, employing [Bel.–Wickramasekera 18', '19: regularity and compactness theory].

Unlike the $g \equiv 0$ case, $\mathcal{F}_{\varepsilon,g} \overset{\varepsilon\to 0}{\to} J_g$ holds in a much weaker sense.

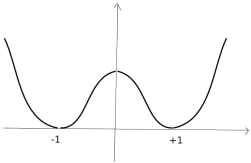
Allen-Cahn / Modica-Mortola (De Giorgi)

Allen-Cahn energy, with parameter $\epsilon \in (0,1)$:

$$\mathcal{E}_\epsilon(u) = \int_N \frac{\epsilon|\nabla u|^2}{2} + \frac{W(u)}{\epsilon}\, d\mathcal{H}^{n+1}, \quad u \in W^{1,2}(N)$$

where

W : R ---> R



$$W : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$$

smooth "double-well potential"

e.g. $W(t) = \frac{1}{4}(1-t^2)^2$.

Allen–Cahn modified for g-prescribed problem

$$\mathcal{F}_{\varepsilon,g}(u) = \mathcal{E}_\varepsilon(u) - \int_N \frac{g}{2}\, u \, d\mathcal{H}^{n+1} - \int_N \frac{g}{2}\, d\mathcal{H}^{n+1}.$$

Euler–Lagrange: $\mathcal{F}'_{\varepsilon,g}(u) = \varepsilon\,\Delta u - \frac{W'(u)}{\varepsilon} = -g/2.$

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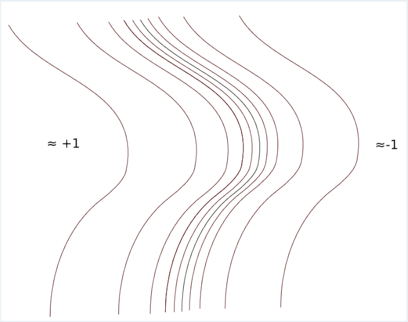
Hope/plan:

- produce u_ε solving $\mathcal{F}'_{\varepsilon,g}(u_\varepsilon) = 0$ by classical minmax (encode the PMC condition "mean curvature prescribed by g" at the ε-level);
- send $\varepsilon \to 0$ and recover a (sufficiently smooth) geometric hypersurface as a limit (in some sense) of u_ε;
- pass to the limit the PMC condition (will need a back-up plan).

Weighted level sets of $u_\epsilon \to$ PMC interface



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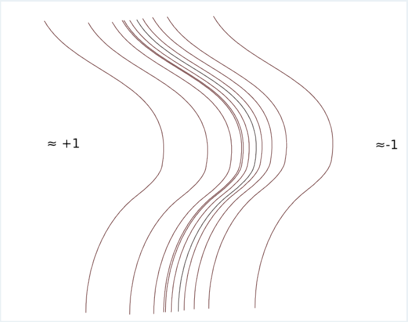

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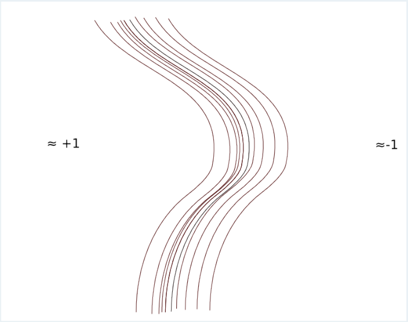

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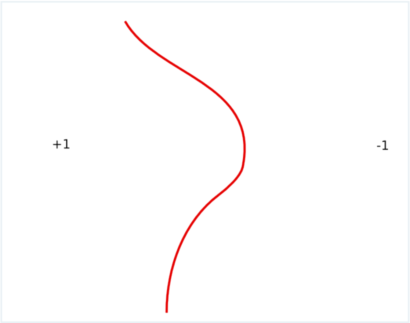


$\mathcal{F}_{\varepsilon,g} \to J_g$?

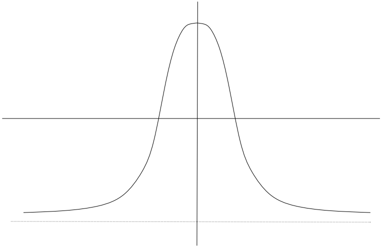
The value of $\mathcal{F}_{\varepsilon,g}$ passes to the limit to become J_g.

Stationarity for $\mathcal{F}_{\varepsilon,g}$ does not become *stationarity* for J_g when $g \neq 0$!

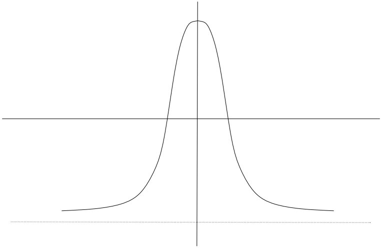
[Hutchinson–Tonegawa '00]
[Röger–Tonegawa '08]

Cancellations and multiplicity

$\forall \varepsilon \quad \exists$ critical point v_ε of $\mathcal{F}_{\varepsilon,g}$ on $\mathbb{R}$ with $g \equiv 1$, whose graph is



larger ε

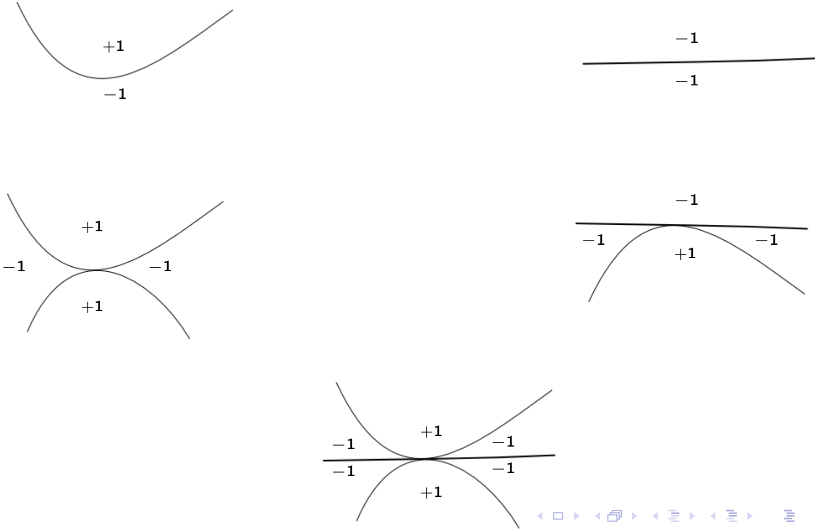


smaller ε

Then $u_\varepsilon(x_1, x_2, x_3) = v_\varepsilon(x_3)$ critical point of $\mathcal{F}_{\varepsilon,1}$; level sets are planes; $\mathcal{E}_\varepsilon(u_\varepsilon)$ concentrates on $\{x_3 = 0\}$ with multiplicity 2 (double transition of the 1-dim profile).

Establish local regularity of limit interface when $g > 0$

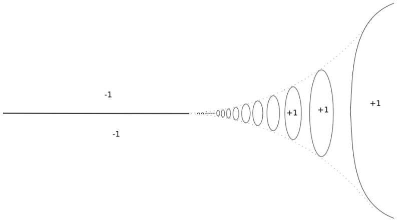
Under Morse index bounds, using [Bel.-Wickramasekera '18 '19]



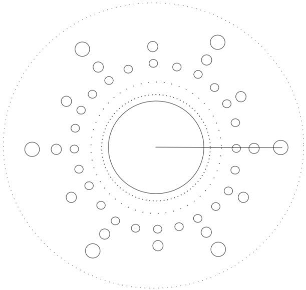
Partial solution when $g > 0$

If the min max leads to an interface that is not only minimal, thanks to the regularity result we can "throw away" any minimal portion and obtain the desired PMC hypersurface. (Take the boundary of the $+1$ phase.)

Would not work without regularity:
E.g. multiplicity-1 PMC portion *with necks*, merging with a multiplicity-2 minimal disk along a singular circle.



vertical cross section



view from above

If we get a minimal-only interface (still $g > 0$):

Back up plan:

If the min max leads to the top-right picture (minimal-only interface) build, by $\mathcal{F}_{\varepsilon,g}$-gradient flow, *another* sequence of (stable) critical points v_ε (of $\mathcal{F}_{\varepsilon,g}$) that will lead to a not-only-minimal interface.

This uses the minmax characterisation of u_ε built earlier.

Recap of the difficulty (still $g > 0$)

Minmax for $\mathcal{F}_{\varepsilon,g}$ produces u_ε s.t. Allen–Cahn energy concentrates onto $\overline{M}$, minimal hypersurface, with even multiplicity, say 2.

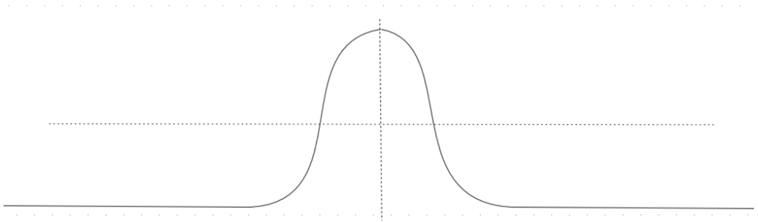
$u_\varepsilon \xrightarrow{L^1} u_\infty \equiv -1$
(no $+1$ phase, so no enclosed volume).

Minmax value (for all small ε) is $\approx 2\mathcal{H}^n(M)$;

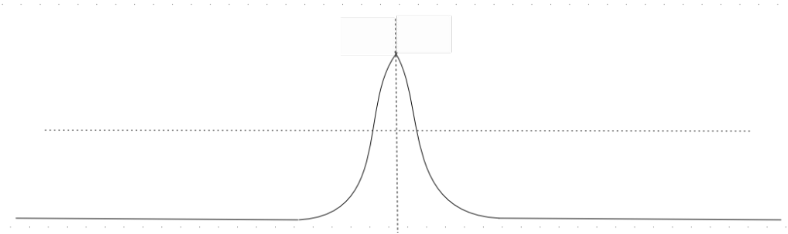
Minmax characterisation $\Rightarrow \nexists$ path (continuous in $W^{1,2}(N)$) that connects the valley points ($\cong -1$ and $\cong +1$) keeping $\mathcal{F}_{\varepsilon,g}$ a fixed amount below $2\mathcal{H}^n(M)$.

We exhibit a specific path (at the ε-level) starting at $\cong -1$; all along it $\mathcal{F}_{\varepsilon,g}$ stays a fixed amount below $2\mathcal{H}^n(M)$.

1-dim. profile to implement multiplicity 2 at the ε-level



Continuously vary multiplicity between 0 and 2 at the ε-level



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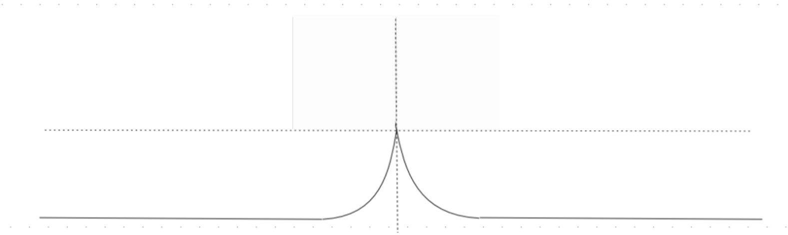

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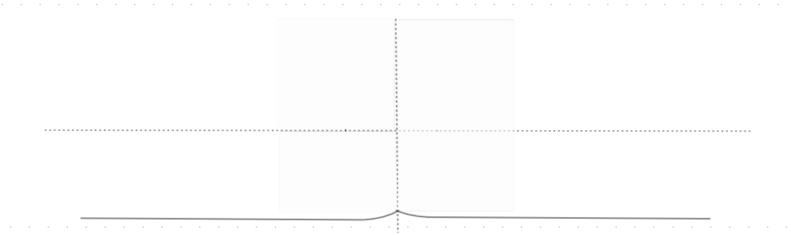
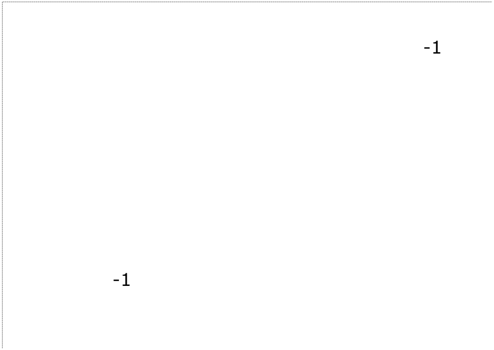


Geometric view of relevant path



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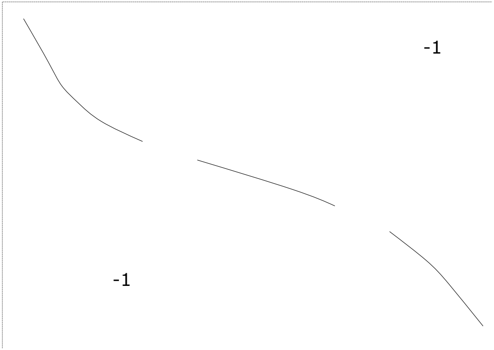

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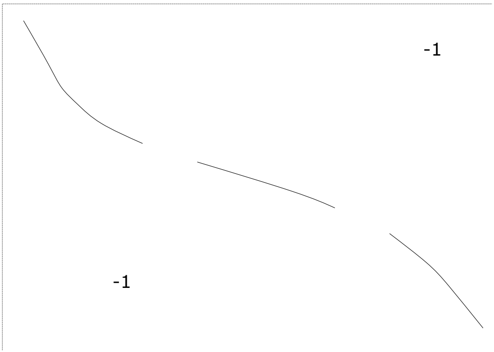

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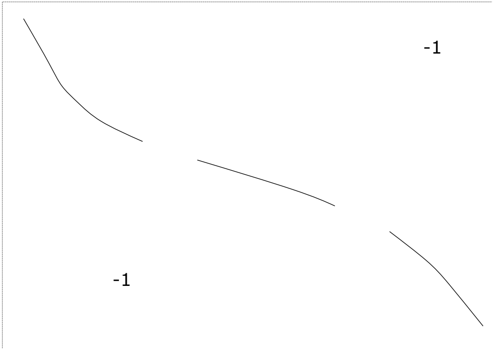

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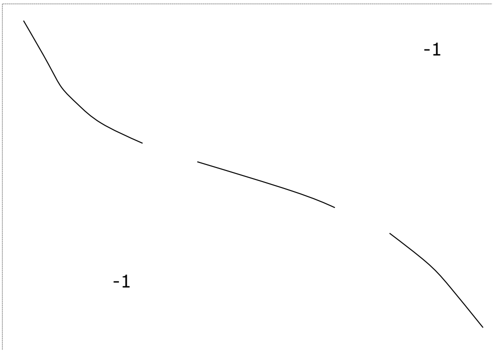

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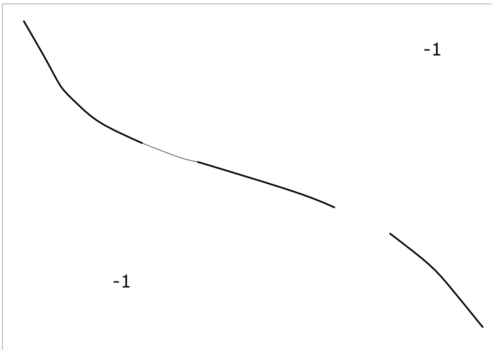

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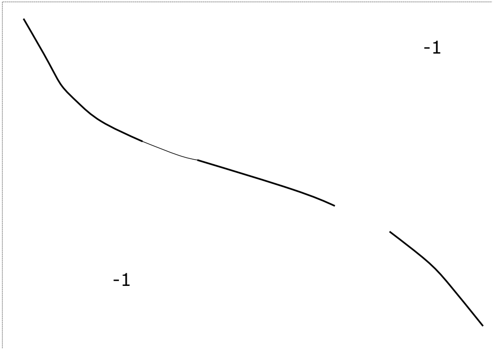

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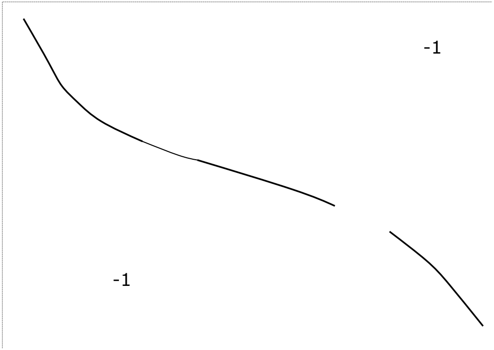

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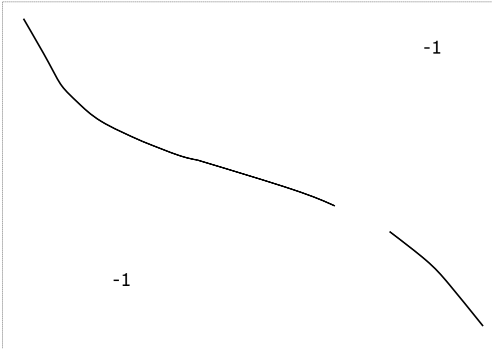

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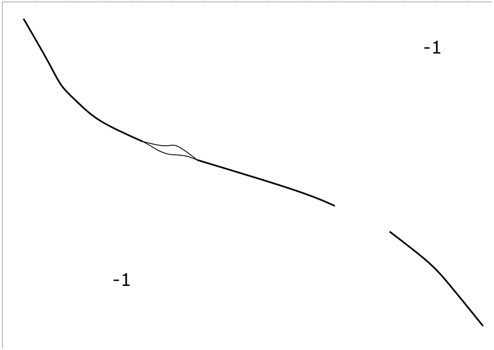

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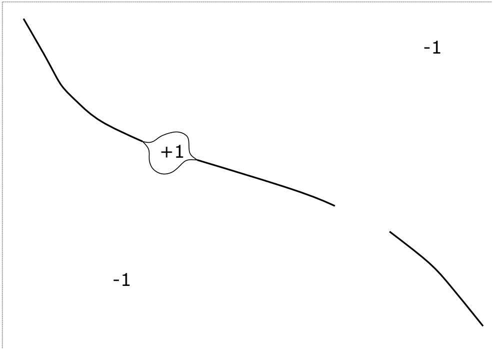
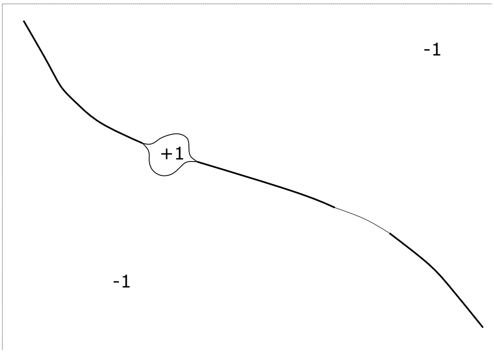
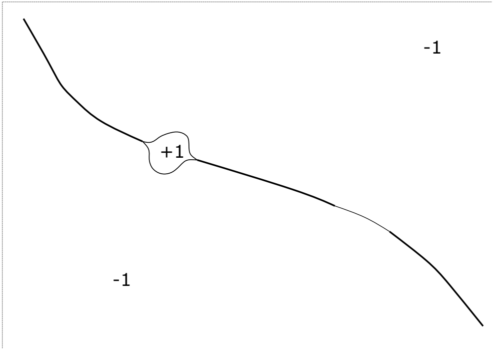


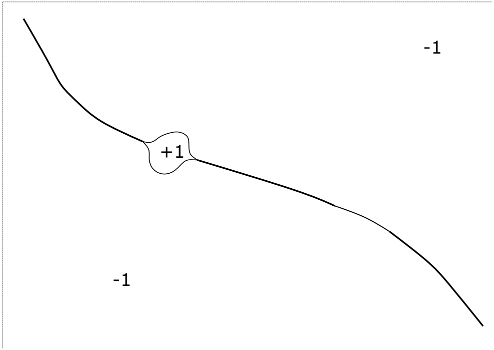
Geometric view of relevant path



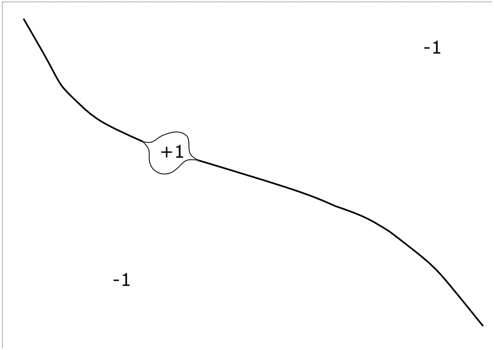
Geometric view of relevant path



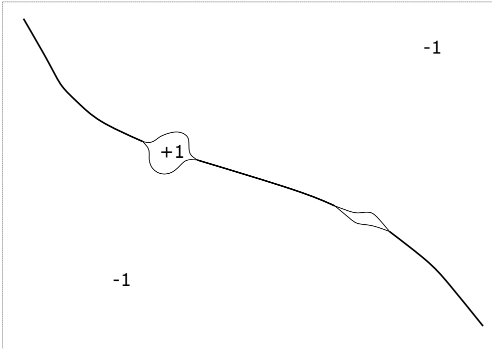
Geometric view of relevant path



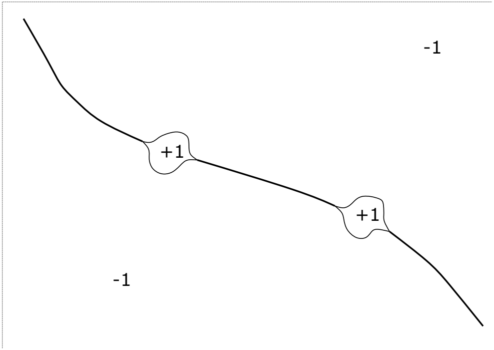
Geometric view of relevant path



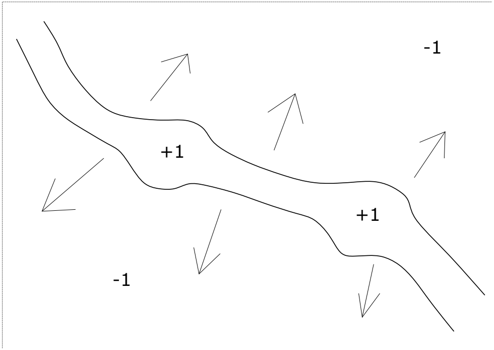
Geometric view of relevant path



Geometric view of relevant path



Geometric view of relevant path



Conclusive argument for $g > 0$

Mean-convexity $\Rightarrow$ flow (for $\nabla \mathcal{F}_{\varepsilon,g}$) converges to a *stable* solution v_ε that is $\approx +1$ on two fixed open regions.

v_ε cannot be everywhere $\approx +1$, because this is the "second valley point" for the minmax — the energy along the path exhibited stays well below the minmax value.

Then (for all ε) $\exists$ v_ε stable critical point of $\mathcal{F}_{\varepsilon,g}$, with both $\{v_\varepsilon \approx +1\}$ and $\{v_\varepsilon \approx -1\}$ non-disappearing in the $\varepsilon \to 0$ limit;

$$\overset{\text{regularity applied to } v_\varepsilon \to v_\infty}{\Longrightarrow}$$

$\partial\{v_\infty = +1\}$ is the desired PMC hypersurface.

$g \geq 0$

Approximate with positive g_j;

pass the interfaces (built for g_j) to the limit, using quantitative (elliptic type) $C^{2,\alpha}$-estimates established in regularity theorem.

No need of constraints on the nodal set $\{g = 0\}$.

Thanks for your attention!