Quasiconvexity in the general growth setting

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Consider

$$\begin{aligned} \text{minimise } \mathcal{F}(u) = \int_\Omega F(x, u, \nabla u)\,\mathrm{d}x, \\ \text{subject to } u = g \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

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Goal 1: Prove existence of minima in Sobolev spaces,
Goal 2: Establish partial $C^{1,\alpha}$-regularity of minima.

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We infer existence via the *Direct Method*, for which we require

1. Take a minimising sequence $\{u_j\}$ for $\mathcal{F}$,
2. Pass to a suitable weak limit $u_j \rightharpoonup u$,
3. Show that the limit map is a minimiser.

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1. a growth condition on F,
2. coercivity of $\mathcal{F}$,
3. semicontinuity of the associated integrand.

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Typically one assumes a p-growth condition

$$|F(x,u,z)| \lesssim |z|^p + 1. \tag{2}$$

However we more generally can replace t^p by an N-function $\varphi(t)$ where

- φ is non-negative, increasing, convex,
- φ satisfies

$$\lim_{t\to 0} \frac{\varphi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\varphi(t)}{t} = +\infty. \tag{3}$$

We also assume the Δ_2-condition, namely $\varphi(2t) \leq C\varphi(t)$.

Examples include

$$\varphi(t) \sim t^p \log t, \tag{4}$$
$$\varphi(t) \sim t^p \log \cdots \log t, \tag{5}$$

for $1 \leq p < \infty$.

However, the linear case $\varphi(t) = t$ is excluded.

Proposition (Meyers-Elcrat 1975, Giaquinta-Modica 1979)

Suppose $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ with $p > 1$ such that for all $B_R(x_0) \subset \Omega$ we have

$$\fint_{B_{R/2}(x_0)} |\nabla u|^p, \mathrm{d}x \leq C \fint_{B_R(x_0)} \frac{|u - (u)_{B_R(x_0)}|^p}{R^p} \,\mathrm{d}x. \tag{6}$$

Then there is $\varepsilon > 0$ such that

$$\nabla u \in L^{p+\varepsilon}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}). \tag{7}$$

This fails for $p = 1$; consider variants of $u(x) = \mathrm{sign}(x)$.

Proposition (Iwaniec 1998, Cianchi-Fusco 1999)

Suppose $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ such that for all $B_R(x_0) \subset \Omega$ we have

$$\fint_{B_{R/2}(x_0)} \varphi(|\nabla u|)\,\mathrm{d}x \leq C \fint_{B_R(x_0)} \varphi\left(\frac{|u-(u)_{B_R(x_0)}|}{R}\right)\mathrm{d}x. \tag{8}$$

Then there is $\kappa > 0$ such that

$$\nabla u \in L^{\varphi^{[\kappa]}}_{\mathrm{loc}}(\Omega, \mathbb{R}^{Nn}), \quad \varphi^{[\kappa]}(t) = \varphi(t)\left(\frac{\varphi(t)}{t}\right)^{\kappa}. \tag{9}$$

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Growth + coercivity assumptions gives a minimising sequence $\{u_j\}$ and a limit

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Is the limit u a minimiser? We need lower semicontinuity of $\mathcal{F}$ in this topology.

Morrey (1952) showed weak* sequential lower semicontinuity in $W^{1,\infty}$ is equivalent to quasiconvexity

$$F(z_0) \leq \fint_\Omega F(z_0 + \nabla \xi)\, \mathrm{d}x \tag{11}$$

for all $z_0 \in \mathbb{R}^{Nn}$, $\xi \in C_c^\infty(\Omega, \mathbb{R}^N)$.

Weak lower semicontinuity for $\mathcal{F}$ in $W^{1,p}$ $(p > 1)$ holds when

1. $F = F(x, u, z)$ is Carathéodory,
2. F satisfies the growth conditions

$$-|z|^r - 1 \lesssim F(x, u, z) \lesssim |z|^p + 1 \tag{12}$$

   with $r < p$,
3. $z \mapsto F(x, u, z)$ is quasiconvex at each (x, u),

Due to Meyers (1965), Acerbi & Fusco (1984), Marcellini (1985) and many others.

Ball & Murat (1984) showed semicontinuity in $W^{1,n}$ fails for $F(z) = \det z$ with

$$u_j(x,y) = j^{-\frac{1}{2}}(1-|y|)^j(\sin jx, \cos jx), \quad (x,y) \in (-\pi,\pi) \times (0,1). \tag{13}$$

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Problem: ∇u_j may concentrate near the boundary, or where F is discontinuous in x.

- Meyers (1965) showed semicontinuity in $W^{1,p}_g(\Omega, \mathbb{R}^N)$ under strong continuity assumptions.
- This allows us to consider for instance

$$F(z) = \frac{1}{n}|z|^n + \det z. \tag{14}$$

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To deduce existence, we only need semicontinuity along *minimising sequences*

$$\{u_j\} \subset W^{1,\varphi}_g(\Omega, \mathbb{R}^N) \tag{15}$$

for which

$$\mathcal{F}(u_j) \to \inf_{v \in W^{1,\varphi}_g(\Omega,\mathbb{R}^N)} \mathcal{F}(v). \tag{16}$$

Assume

1. $F = F(x, u, z)$ is Carathéodory,
2. F satisfies the growth condition

$$|F(x, u, z)| \lesssim 1 + \varphi(|z|), \tag{17}$$

3. $z \mapsto F(x, u, z)$ is quasiconvex at each (x, u),
4. there is $\nu > 0$ and $f \colon \mathbb{R}^{Nn} \to \mathbb{R}$ such that $f(z) \leq F(x, u, z)$ and

$$f - \nu\varphi(|\cdot|) \text{ is quasiconvex at } 0. \tag{18}$$

Proposition

Let $\{u_j\}$ be a minimising sequence, there is a sequence $\{v_j\}$ such that

$$u_j - v_j \overset{*}{\rightharpoonup} 0, \quad \text{and} \quad \mathcal{F}(v_j) \leq \mathcal{F}(u_j) \tag{19}$$

such that

$$\{\varphi(|\nabla v_j|)\} \text{ is uniformly integrable.} \tag{20}$$

Here $\Omega \subset \mathbb{R}^n$ is bounded open and $g \in W^{1,\varphi}(\mathbb{R}^n, \mathbb{R}^N)$.

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Corollary

Let $\{u_j\}$ be a minimising sequence for $\mathcal{F}$ and $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\varphi}$, then

$$\mathcal{F}(u) \leq \liminf_{j\to\infty} \mathcal{F}(u_j). \tag{21}$$

Hence one can show the existence of minimisers in $W^{1,\varphi}_g(\Omega, \mathbb{R}^n)$.

Step 1: By Ekeland's variational principle we have $\{v_j\} \subset W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$ such that

$$\mathcal{F}(v_j) \leq \mathcal{F}(w) + \int_\Omega |\nabla v_j - \nabla w| \,\mathrm{d}x \tag{22}$$

for all j and $w \in W^{1,\varphi}_g(\Omega, \mathbb{R}^N)$.

Step 2: Then one can infer a Caccioppoli inequality

$$\fint_{B_{R/2}(x_0)} \varphi(|\nabla v_j|) \leq C \fint_{B_R(x_0)} 1 + \varphi(|\nabla g|) + \varphi\left(\frac{|v_j - (v_j)_{B_R(x_0)}|}{R}\right) \mathrm{d}x \tag{23}$$

and also up to the boundary.

Proposition (Iwaniec 1998, Cianchi-Fusco 1999)

Suppose $u \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ such that for all $B_R(x_0) \subset \Omega$ we have

$$\fint_{B_{R/2}(x_0)} \varphi(|\nabla u|)\,\mathrm{d}x \leq C \fint_{B_R(x_0)} \varphi\left(\frac{|u-(u)_{B_R(x_0)}|}{R}\right)\mathrm{d}x. \tag{8}$$

Then there is $\kappa > 0$ such that

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Step 3: A variant of Gehring's lemma gives

$$\sup_j \int_\Omega \theta \circ \varphi(|\nabla v_j|)\,\mathrm{d}x < \infty, \tag{24}$$

where $\theta(t)/t \to \infty$ as $t \to \infty$.

This gives uniform φ-integrability.

Thank you for listening! Any questions?

On the regularity side one seeks ε-regularity results of following form:

For each $M \geq 0$, there is $\varepsilon_M > 0$ such that if

$$|(\nabla u)_{B_R(x_0)}| \leq M, \quad \fint_{B_R(x_0)} \varphi_1(|\nabla u - (\nabla u)_{B_R(x_0)}|)\,\mathrm{d}x < \varepsilon_M, \tag{25}$$

then u is $C^{1,\alpha}$ in $B_{R/2}(x_0)$.

Considered by Evans (1986), Acerbi-Fusco (1987), Carozza-Fusco-Mingione (1998), Diening-Lengeler-Stroffolini-Verde (2006), Gmeineder-Kristensen (2019), and many others. . .

We can establish results of the above type when:

- $F = F(z)$ and $\varphi \in \Delta_2 \cap \nabla_2$.
- $F = F(z)$ and $\varphi(t) \sim t \log \cdots \log t$.

Case $F = F(x, u, z)$ is more complicated...

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- The conjugate function $\varphi^* \in \Delta_2$,
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Rough idea: Δ_2 and ∇_2 conditions give polynomial control from above and below.