Non-uniqueness of Leray solutions of the forced Navier-Stokes equations [1]

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[1] Joint with D. Albritton and M. Colombo

The Navier-Stokes system

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = f \\ \operatorname{div} u = 0 \\ u(\cdot, 0) = u_0 \end{cases} \qquad \text{on } \mathbb{R}^3 \times [0, T] \qquad \text{(NS)}$$

Millennium problem

Assume $u_0 \in C_c^\infty$ and $f = 0$. Is there a global smooth solutions to (NS)?

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Leray-Hopf solutions

Let $u_0 \in L^2$, $f \in L^1_t L^2_x$.

- [Leray '34], [Hopf '51]: Global solutions to (NS) in the class

$$u \in L^\infty_t L^2_x \cap L^2_t H^1_x \,;$$

- $u(\cdot, 0) = u_0$;
- Energy inequality:

$$\frac{1}{2}\int |u(x,t)|^2 \,\mathrm{d}x + \int_0^t \int |\nabla u(x,s)|^2 \,\mathrm{d}x \,\mathrm{d}s \leq \frac{1}{2}\int |u_0(x)|^2 \,\mathrm{d}x + \int_0^t \int f(x,s) \cdot u(x,s) \,\mathrm{d}x \,\mathrm{d}s \,.$$

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Properties of Leray-Hopf solutions

- Suitability condition:

$$(\partial_t - \Delta)\frac{1}{2}|u|^2 + |\nabla u|^2 + \operatorname{div}\left(\left(\frac{1}{2}|u|^2 + p\right)u\right) \leq f \cdot u\,.$$

- Partial regularity: If $f \in L^{5/2+}_{x,t}$ then

$$\mathcal{P}^1(\text{singular set}) = 0\,, \quad \left(\mathscr{H}^{1/2}(\text{singular times}) = 0\right)\,,$$

[Caffarelli-Kohn-Nirenberg '82].

- Weak-strong uniqueness: Leray-Hopf solutions agree with strong solutions.

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The uniqueness problem

- The program of Jia, Sverak, and Guillod:
  - Bifurcation from large self-similar solutions.
    Need to prove the existence of an unstable self-similar background [Jia-Sverak '14, '15].
  - Numerical evidence of instability [Guillod-Sverak '17].

- Convex integration method: After [De Lellis, Szekelyhidi '09, '13]
  - Non-uniqueness of weak solutions to (NS) in

    $$u \in C_t H_x^\beta\,, \quad \text{for some } \beta > 0\,,$$

    [Buckmaster-Vicol '19].
  - Non-uniqueness in $L_t^q L_x^\infty$, $q < 2$ and $C_t L_x^p$, $p < 2$. [Cheskidov-Luo '20].

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Main results

Theorem (Albritton-B.-Colombo '21)

There exist u and $\bar{u}$, two distinct suitable Leray-Hopf solutions to (NS) *on $\mathbb{R}^3$ with identical body force $f \in L^1_t L^2_x$ and $u(\cdot, 0) = \bar{u}(\cdot, 0) = 0$.*

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The same conclusion holds in bounded domains and in $\mathbb{T}^3$.

Self-similar structure

- There exists a div-free velocity field $\bar{U} \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ s.t.

$$\bar{u}(x,t) = \frac{1}{\sqrt{t}}\bar{U}\left(\frac{x}{\sqrt{t}}\right).$$

$$\bar{u} \in C^\infty(\mathbb{R}^3 \times (0,T)) \cap C^0([0,T]; L^1 \cap L^{3-}).$$

- There exists $F \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$f(x,t) = \frac{1}{t^{3/2}}F\left(\frac{x}{\sqrt{t}}\right).$$

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Similarity variables

Let u be a solution to (NS) with body force f.

- Change of variables: $\xi = x/\sqrt{t}$, $\tau = \log(t) \in (-\infty, T)$

$$u(x,t) = \frac{1}{\sqrt{t}} U(\xi,\tau)\,,$$

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- (NS) in similarity variables: $(\xi,\tau) \in \mathbb{R}^3 \times (-\infty, T)$

$$\partial_\tau U - \frac{1}{2}(1 + \xi\cdot\nabla)U - \Delta U + U\cdot\nabla U + \nabla P = F$$

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The second solution

- We think of $\bar{U}$ as a stationary solutions to (NS) is similarity variables with body force F

$$-\frac{1}{2}(1+\xi\cdot\nabla)\bar{U}-\Delta\bar{U}+\bar{U}\cdot\nabla\bar{U}+\nabla P=F\,.$$

- Instability in similarity variables $\implies$ non-uniqueness.
- If $\bar{U}$ is (linear) unstable, we make the Ansatz

$$U=\bar{U}+V=\bar{U}+U^{\mathrm{lin}}+U^{\mathrm{per}}\,,$$

  where
  - U^{lin} solves the linearized (NS) equations around $\bar{U}$
  - U^{per} is a small perturbation

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- $U = \bar{U} + V$ solves (NS) iff

$$\partial_\tau V = -\mathbb{P}(\bar{U} \cdot \nabla V + V \cdot \nabla \bar{U}) + \Delta V + \frac{1}{2}(1 + \xi \cdot \nabla)V - \mathbb{P}(V \cdot \nabla V)$$
$$= \mathcal{L}_{ss} V - \mathbb{P}(V \cdot \nabla V), \qquad (\xi, \tau) \in \mathbb{R}^3 \times (-\infty, T).$$

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$$-\mathcal{L}_{ss} V = -\frac{1}{2}(1 + \xi \cdot \nabla)V - \Delta V + \mathbb{P}(\bar{U} \cdot \nabla V + V \cdot \nabla \bar{U}).$$

- Goal: We look for a nontrivial solution V which decays at $\tau \to -\infty$.

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The spectral problem

- Linearized equations: We drop the non-linear term $\mathbb{P}(V \cdot \nabla V)$,

$$\begin{cases} \partial_\tau V = \mathcal{L}_{ss} V \\ V(\xi, \tau) \to 0 \quad \text{as } \tau \to -\infty \end{cases}$$

- Spectral problem: We look for an unstable eigenvalue of $\mathcal{L}_{ss}$, i.e.

$$\lambda \in \mathbb{C}\,, a := \mathrm{Re}\lambda > 0\,, \qquad \eta \in C^\infty \cap H^k \;\; \forall k > 0$$

such that

$$\mathcal{L}_{ss}\eta = \lambda\eta\,.$$

$$U^{\mathrm{lin}}(\xi, \tau) = \mathrm{Re}(e^{\lambda\tau}\eta(\xi))\,, \quad \tau \in \mathbb{R}\,.$$

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$$\begin{cases} \partial_\tau V = \mathcal{L}_{ss} V \\ V(\xi, \tau) \to 0 \quad \text{as } \tau \to -\infty \end{cases}$$

- Spectral problem: We look for an unstable eigenvalue of $\mathcal{L}_{ss}$, i.e.

$$\lambda \in \mathbb{C}\,, a := \mathrm{Re}\lambda > 0\,, \qquad \eta \in C^\infty \cap H^k \ \ \forall k > 0$$

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$$\mathcal{L}_{ss}\eta = \lambda\eta\,.$$

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$$\partial_\tau V = \mathcal{L}_{ss} V - \mathbb{P}(V \cdot \nabla V) \, .$$

We make the Ansatz:

$$V = U^{\text{lin}} + U^{\text{per}} \, , \quad |U^{\text{per}}(\xi, \tau)| \sim e^{2\text{Re}\lambda\tau} \, .$$

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Theorem (Albritton-B.-Colombo)

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- 2d-vorticity formulation: $\omega(x) = \operatorname{curl} u(x)$, $x \in \mathbb{R}^2$

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- Shear flows: $x = (x_1, x_2) \in \mathbb{R}^2$,

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There exists a smooth decaying vortex

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such that the linear operator $\mathcal{L}_{st} : D(\mathcal{L}_{st}) \subset L^2_m(\mathbb{R}^2) \to L^2_m(\mathbb{R}^2)$*,* $m \geq 2$*,*

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Sharpness of Yudovich class

Theorem (Vishik'18, ABCDGJK'21)

For every $p \in (2, \infty)$, there exist two distinct finite-energy weak solutions u and $\bar{u}$ of the $2d$-Euler equations with identical body force f such that

- $\omega, \bar{\omega} \in L_t^\infty (L_x^p \cap L_x^1)$;
- $f \in L_t^1 L_x^2$ *and* $\mathrm{curl} f \in L_t^1 (L_x^p \cap L_x^1)$.

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Theorem (Albritton-B.-Colombo)
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Axisymmetric-no-swirl structure

$$x = (r\cos\theta, r\sin\theta, z) \in \mathbb{R}^3$$

$$U = U^r(r,z)e_r + U^z(r,z)e_z\,,$$

$$\mathrm{curl}U = -\Omega(r,z)e_\theta\,.$$

- We assume $\bar{U} = \bar{U}^r(r,z)e_r + \bar{U}^z(r,z)e_z$.
- The space of axisymmetric-no-swirl velocity fields is invariant under the action of $\mathcal{L}_{st}$.

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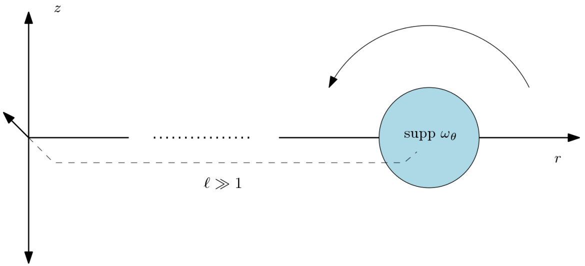
where $\omega \in L^2(\mathbb{R}_+ \times \mathbb{R})$.



3D Instability: Vortex ring

We lift Vishik's unstable vortex

$$\bar{u}(x) = \zeta(r)x^{\perp}, \quad \bar{\omega}(x) = g(r), \quad x \in \mathbb{R}^2, r = |x|,$$

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- Truncation step: $\bar{u}_R(x) := \varphi_R(r)\zeta(r)x^{\perp}$ is unstable provided $R >> 1$.
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$$\bar{u}_R(x_1, x_2) = (\bar{u}_R^1(x_1, x_2), \bar{u}_R^2(x_1, x_2))$$

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$$\partial_r^2 \psi + \frac{1}{r}\partial_r \psi - \frac{1}{r^2}\psi + \partial_z^2 \psi = \omega \quad \text{in } \mathbb{R}_+ \times \mathbb{R}\,,$$

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