Gravitational Collapse of Self-similar Stars

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Newtonian stars

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- As gas burns, balance shifts;
- Possible collapse? Supernova?

Figure: Image credit: R.J. Hall

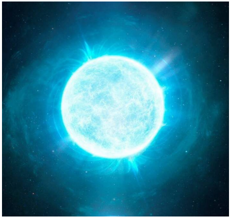
Euler-Poisson equations

The Euler-Poisson equations of gas dynamics with Newtonian gravity:

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\big) + \nabla_{\mathbf{x}} p(\rho) = -\rho \nabla \Phi, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \\ \Delta \Phi = 4\pi \rho, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3. \end{cases} \tag{1}$$

ρ is density, $\mathbf{u}$ is velocity, p is pressure, Φ is gravitational potential.
We assume the equation of state

$$p = p(\rho) = \rho^\gamma, \quad \gamma \in \big(1, \frac{4}{3}\big).$$



Collapse

Collapse is the formation of a *singularity* at the origin, i.e.

$$\rho(t,0) \to \infty \quad \text{as} \quad t \to 0- \, .$$

- For $\gamma > \frac{4}{3}$, no finite mass and energy collapse possible.
- For $\gamma = \frac{4}{3}$, Goldreich–Weber collapse - unsuitable model for outer core.

Previous Results

Classical and numerical work

- Taylor, Von Neumann, Sedov, Güderley '40s: study implosion and explosion for Euler equations;
- Larson–Penston '69: numerical solution for $\gamma = 1$;
- Hunter '77: family of numerical solutions for $\gamma = 1$;
- Yahil '83: numerical solutions for $\gamma \in [\frac{6}{5}, \frac{4}{3})$;
- Maeda–Harada '01: numerical evidence towards mode stability of Larson–Penston.

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Recent works

- Merle–Raphaël–Rodnianski–Szeftel '22: existence of a imploding self-similar solutions for Euler;
- Guo–Hadzic–Jang '21: construction of LP solution.

Self-similar singularity formation (Type I)

Self-similarity and singularities interact in a wide range of problems.

- Stellar collapse;
- Formation/expansion of shock waves (Buckmaster–Shkoller–Vicol '20,...);
- Shock reflection (Chen–Feldman '18);
- Bacterial growth;
- Geometric wave equations (Costin–Donninger–Glogic '17,...);
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Key Features:

- Non-linearity;
- Intertwining of spatial and time scales;
- Good initial data leads to badly behaved solutions!

UCL

Scaling

Let $\rho = \rho(t, r)$, $\mathbf{u} = u(t, r)\frac{\mathbf{x}}{|\mathbf{x}|}$, $r = |\mathbf{x}|$, solve Euler-Poisson, $\lambda > 0$. Then

$$\rho_\lambda(t, r) = \lambda^{-\frac{2}{2-\gamma}}\rho\left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\right), \quad u_\lambda(t, r) = \lambda^{-\frac{\gamma-1}{2-\gamma}}u\left(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\right)$$

is also a solution. (NB: This is a *unique* scaling!)

Scaling and Self-similarity

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Self-similarity

We define a *self-similar* variable

$$y = \frac{r}{(-t)^{2-\gamma}},$$

and search for

$$\rho(t,r) = (-t)^{-2}\tilde{\rho}(y), \quad u(t,r) = (-t)^{1-\gamma}\tilde{u}(y).$$

ODE system

Defining a convenient variable $\omega(y) = \tilde{u}(y)/y + 2 - \gamma$, self-similar Euler-Poisson becomes

$$\begin{aligned} \tilde{\rho}' &= \frac{y\tilde{\rho}h(\tilde{\rho},\omega)}{\gamma\tilde{\rho}^{\gamma-1} - y^2\omega^2}, \\ \omega' &= \frac{4 - 3\gamma - 3\omega}{y} - \frac{y\omega h(\tilde{\rho},\omega)}{\gamma\tilde{\rho}^{\gamma-1} - y^2\omega^2}, \end{aligned} \tag{2}$$

where $h(\tilde{\rho},\omega)$ is a quadratic function.

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Definition (Sonic point)

Let $(\tilde{\rho}(\cdot),\omega(\cdot))$ be a C^1-solution to the self-similar Euler-Poisson system on the interval $(0,\infty)$. A point $y_* \in (0,\infty)$ such that

$$G(y,\tilde{\rho},\omega) := \gamma\tilde{\rho}^{\gamma-1}(y_*) - y_*^2\omega^2(y_*) = 0$$

is called a *sonic point*.

UCL

Initial/boundary conditions

For a regular solution, we require

$$\tilde{\rho}(0) > 0, \quad \omega(0) = \frac{4-3\gamma}{3},$$

$$\tilde{\rho}(y) \sim y^{-\frac{2}{2-\gamma}} \text{ as } y \to \infty, \quad \lim_{y\to\infty} \omega(y) = 2-\gamma.$$

NB: this forces the existence of a sonic point!

Theorem

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Theorem (Guo–Hadzic–Jang–S. '21)

For each $\gamma \in (1, \frac{4}{3})$, there exists a global, real-analytic solution $(\tilde{\rho}, \omega)$ of self-similar Euler-Poisson with a single sonic point y_* such that:

$$\tilde{\rho}(y) > 0 \text{ for all } y \in [0,\infty), \quad -\frac{2}{3}y < \tilde{u}(y) < 0 \text{ for all } y \in (0,\infty).$$

In addition, both ρ and ω are strictly monotone:

$$\tilde{\rho}'(y) < 0 \text{ for all } y \in (0,\infty), \quad \omega'(y) > 0 \text{ for all } y \in (0,\infty).$$

Overview of key difficulties

Regularity

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Non-autonomous system

Non-autonomous forces evolving phase portrait. No fixed phase portrait analysis for invariant regions.

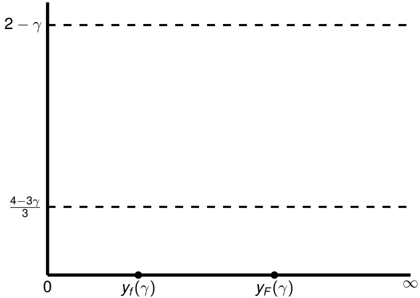
Overview of Strategy

Two explicit solutions

Far-field solution (ρ_f, ω_f) and Friedman solution (ρ_F, ω_F):

$$(\rho_f(y), \omega_f(y)) = (k_\gamma y^{-\frac{2}{2-\gamma}}, 2-\gamma), \qquad (\rho_F(y), \omega_F(y)) = (\frac{1}{6\pi}, \frac{4}{3} - \gamma).$$

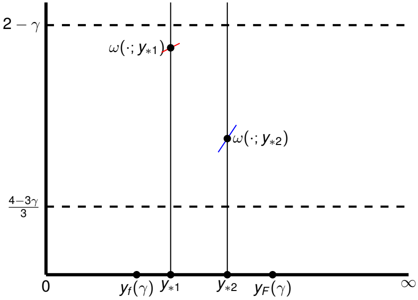
Sonic points at $y_f(\gamma) < y_F(\gamma)$.



Overview of Strategy

Proposition (Local Solution)

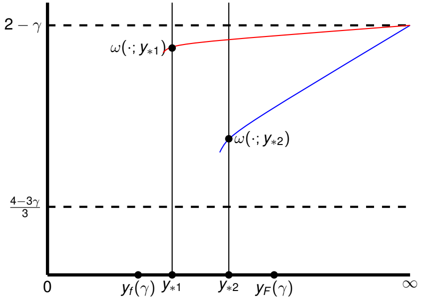
For all $\gamma \in (1, \frac{4}{3})$, there exists $\nu > 0$ such that for all $y_* \in [y_f(\gamma), y_F(\gamma)]$, there exists an analytic solution $(\rho(\cdot; y_*), \omega(\cdot; y_*))$ to self-similar Euler-Poisson on $(y_* - \nu, y_* + \nu)$ with a single sonic point at y_*.



Overview of Strategy

Lemma (Solving to the right)

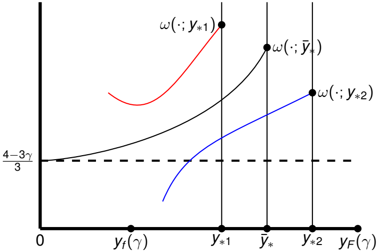
For each $\gamma \in (1, \frac{4}{3})$*, each* $y_* \in [y_f(\gamma), y_F(\gamma)]$*, the local solution* $(\rho(\cdot; y_*), \omega(\cdot; y_*))$ *obtained by Taylor expansion extends globally to the right on* $[y_*, \infty)$*, remains supersonic, and satisfies the asymptotic boundary conditions.*



Overview of Strategy

Aim: Find $\bar{y}_*$ such that local solution $(\rho(\cdot;\bar{y}_*), \omega(\cdot;\bar{y}_*))$ extends smoothly to $y = 0$. Look for solution with

$$\frac{4}{3} - \gamma \leq \omega(y;\bar{y}_*) < 2 - \gamma, \qquad \lim_{y\to 0} \omega(y;\bar{y}_*) = \frac{4}{3} - \gamma.$$



Future Programme

Linear Stability

- Appropriate self-similar coordinates;
- Non-self-adjoint problem (complex eigenvalues);
- Sonic degeneracy and issues with dissipativity (monotonicity).

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Future directions

- Non-linear stability;
- Einstein-Euler (relativistic self-similar fluid implosion) and its stability (cf. Guo–Hadžić–Jang ’21).
- Continuation and expansion?

Thank you!