MAT syllabus

Laws of logarithms and exponentials. Solution of the equation $a^x = b$.

Revision

- $a^m a^n = a^{m+n}$ for any positive real number $a$ and any real numbers $m$ and $n$.
- $(a^m)^n = a^{mn}$ for any positive real number $a$ and any real numbers $m$ and $n$.
- $a^{-n} = \frac{1}{a^n}$ for any positive real number $a$ and any real number $n$.
- $a^0 = 1$ for any non-zero real number $a$.
- The solution to $a^x = b$ where $a$ and $b$ are positive numbers (with $a \neq 1$) is $\log_a(b)$. In this expression, the number $a$ is called the base of the logarithm.
- $\log_a(x)$ is a function of $x$ which is defined when $x > 0$. Like with $\sin x$, sometimes the brackets are omitted if it’s clear what the function is being applied to, so we might write $\log_a x$.
- $\log_a x$ doesn’t repeat any values; if $\log_a x = \log_a y$ then $x = y$.
- Note the special case $\log_a a = 1$ because $\log_a a$ is the solution $x$ to the equation $a^x = a$, and that solution is 1.
- In fact, $\log_a(a^x) = x$.
- In that sense, the logarithm function is the inverse function for $y = a^x$.
- $a^{\log_a x} = x$.
- $\log_a(xy) = \log_a(x) + \log_a(y)$.
- $\log_a(x^k) = k \log_a x$ including $\log_a \frac{1}{x} = -\log_a x$.
- There’s a mathematical constant called $e$, which is just a number (it’s about 2.7).
- $e^x$ is called the exponential function.
- The laws of indices and laws of logarithms above hold when the base $a$ is equal to $e$.
- $\log_e x$ is sometimes written as $\ln x$ and the function is sometimes called the natural logarithm.
Warm-up

1. Simplify \((2^3)^4\) and \((2^4)^3\) and \(2^4 \cdot 2^3\) and \(2^3 \cdot 2^4\).
2. Solve \(x^{-2} + 4x^{-1} + 3 = 0\).
3. Solve \(\log_x(x^2) = x^3\).
4. Solve \(\log_{x+5}(6x + 22) = 2\).
5. Simplify \(\log_{10} 3 + \log_{10} 4\) into a single term.
6. If we write \(a = \ln 2\) and \(b = \ln 5\), then write the following in terms of \(a\) and \(b\).
   \[\ln 1024, \quad \ln 40, \quad \ln \sqrt[5]{\frac{2}{5}}, \quad \ln \frac{1}{10}, \quad \ln 1.024.\]
7. Expand \((e^x + e^{-x})(e^y - e^{-y}) + (e^x - e^{-x})(e^y + e^{-y})\).
8. Expand \((e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})\).
9. Solve \(2^x = 3\). Solve \(0.5^x = 3\). Solve \(4^x = 3\).
10. For which values of \(x\) (if any) does \(1^x = 1\)? For which values of \(x\) (if any) does \(1^x = 3\)?
11. For what values of \(b\) (if any) does \(0^b = 0\)? For what values of \(a\) (if any) does \(a^0 = 0\)?

For solutions see [www.maths.ox.ac.uk/r/matlive](http://www.maths.ox.ac.uk/r/matlive)
MAT questions

MAT 2015 Q1H
How many distinct solutions does the following equation have?

\[ \log_{x^2+2}(4 - 5x^2 - 6x^3) = 2 \]

(a) None, (b) 1, (c) 2, (d) 3, (e) Infinitely many.

Hint: that’s a scary logarithm! How can we get rid of it?

MAT 2017 Q1I
Let \( a, b, c > 0 \) and \( a \neq 1 \). The equation

\[ \log_b ((b^x)^x) + \log_a \left( \frac{c^x}{b^x} \right) + \log_a \left( \frac{1}{b} \right) \log_a (c) = 0 \]

has a repeated root when

(a) \( b^2 = 4ac \), (b) \( b = \frac{1}{a} \), (c) \( c = \frac{b}{a} \), (d) \( c = \frac{1}{b} \), (e) \( a = b = c \).

Hint: there’s a lot of simplifying to do here before this turns into a polynomial.
MAT 2013 Q1F
Three positive numbers $a$, $b$, $c$, satisfy

$$\log_b a = 2, \quad \log_b (c - 3) = 3, \quad \log_a (c + 5) = 2.$$ 

This information

(a) specifies $a$ uniquely.
(b) is satisfied by exactly two values of $a$
(c) is satisfied by infinitely many values of $a$
(d) is contradictory.

Hint: this is secretly a system of equations for $a$, $b$, and $c$. The option “specifies $a$ uniquely” would be true if there is exactly one value of $a$ that works in these equations, and perhaps one or more solutions for $b$ and $c$. The option “is contradictory” would be true if there are no solutions for at least one of the variables.

MAT 2013 Q1J
For a real number $x$ we denote by $[x]$ the largest integer less than or equal to $x$. Let $n$ be a natural number. The integral

$$\int_0^n [2^x] \, dx$$

equals

(a) $\log_2 ((2^n - 1)!)$, (b) $n2^n - \log_2 ((2^n)!)$, (c) $n2^n$, (d) $\log_2 ((2^n)!)$,

where $k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer $k$.

Hint: split the integral up into different regions where $2^x$ takes values in between different whole numbers.

For solutions see www.maths.ox.ac.uk/r/matlive
Extension

The following material is included for your interest only, and not for MAT preparation.

This isn’t on the MAT syllabus, but $\ln x$ plays a special role in calculus. It’s the indefinite integral of $x^{-1}$. Let’s explore that. First, here’s a quick reminder that we can’t integrate $x^{-1}$ with our normal rule for integrating $x^n$, which would give $\frac{x^{n+1}}{n+1}$, because we can’t divide by $n+1$ if $n = -1$. But the area under the graph $y = x^{-1}$ from, say, $x = 1$ to $x = 2$ is just some real number! It’s perhaps surprising that it’s $\ln 2$.

To get an idea of the link between integrating $x^{-1}$ and $\ln x$, let’s write down the problem we’re trying to solve;

Find a function $y(x)$ such that $\frac{dy}{dx} = \frac{1}{x}$.

Here’s a trick – we can flip both sides of this equation the other way up to get

$$\frac{dx}{dy} = x$$

This sort of manipulation is definitely not on the MAT syllabus, and it’s really not obvious that the inverse of $\frac{dy}{dx}$ should be $\frac{dx}{dy}$ because the derivative there is not really a fraction, it’s more like a notation for a limit, but trust me, this operation does actually work.

Now if we squint at this new equation, it’s telling us something about the derivative of $x$ in terms of $y$ (if it helps, switch the $x$ in this equation for something that looks like a fancy $y$ and switch the $y$ for something that looks like a fancy $x$). It says that when we differentiate $x$ with respect to $y$, we get $x$ back. That’s exactly what $e^y$ does! So we can integrate and write $x = e^y$. Rearranging, this gives $y = \ln x$. Magic!

(Technical note - we could in fact have chosen $x = Ae^y$ for any constant $A$, and then we’d have got $y = \ln x + c$ for some constant $c$. This is exactly what we should have expected from the original problem – don’t forget the constant of integration!)

Exponentials can also be used to define two new functions;

$$f(x) = \frac{1}{2} \left( e^x + e^{-x} \right) \quad \text{and} \quad g(x) = \frac{1}{2} \left( e^x - e^{-x} \right).$$

These have lots of nice properties which you can check, such as $f'(x) = g(x)$ and $g'(x) = f(x)$ and $f(x)^2 - g(x)^2 = 1$. They are called hyperbolic trigonometric functions, and they’re usually written as $f(x) = \cosh x$ and $g(x) = \sinh x$. You can probably guess what $\tanh x$ is.