MAT syllabus

Laws of logarithms and exponentials. Solution of the equation $a^x = b$.

Revision

- $a^m a^n = a^{m+n}$ for any positive real number a and any real numbers m and n.
- $(a^m)^n = a^{mn}$ for any positive real number a and any real numbers m and n.
- $a^{-n} = \dfrac{1}{a^n}$ for any positive real number a and any real number n.
- $a^0 = 1$ for any non-zero real number a.
- The solution to $a^x = b$ where a and b are positive numbers (with $a \neq 1$) is $\log_a(b)$. In this expression, the number a is called the base of the logarithm.
- $\log_a(x)$ is a function of x which is defined when $x > 0$. Like with $\sin x$, sometimes the brackets are omitted if it's clear what the function is being applied to, so we might write $\log_a x$.
- $\log_a x$ doesn't repeat any values; if $\log_a x = \log_a y$ then $x = y$.
- Note the special case $\log_a a = 1$ because $\log_a a$ is the solution x to the equation $a^x = a$, and that solution is 1.
- In fact, $\log_a(a^x) = x$.
- In that sense, the logarithm function is the inverse function for $y = a^x$.
- $a^{\log_a x} = x$.
- $\log_a(xy) = \log_a(x) + \log_a(y)$.
- $\log_a(x^k) = k \log_a x$ including $\log_a \dfrac{1}{x} = -\log_a x$.
- There's a mathematical constant called e, which is just a number (it's about 2.7).
- e^x is called the exponential function.
- The laws of indices and laws of logarithms above hold when the base a is equal to e.
- $\log_e x$ is sometimes written as $\ln x$ and the function is sometimes called the natural logarithm.

Warm-up

1. Simplify $(2^3)^4$ and $(2^4)^3$ and $2^4 2^3$ and $2^3 2^4$.
2. Solve $x^{-2} + 4x^{-1} + 3 = 0$.
3. Solve $\log_x(x^2) = x^3$.
4. Solve $\log_{x+5}(6x + 22) = 2$.
5. Simplify $\log_{10} 3 + \log_{10} 4$ into a single term.
6. If we write $a = \ln 2$ and $b = \ln 5$, then write the following in terms of a and b.

$$\ln 1024, \quad \ln 40, \quad \ln \sqrt{\frac{2}{5}}, \quad \ln \frac{1}{10}, \quad \ln 1.024.$$

7. Expand $(e^x + e^{-x})(e^y - e^{-y}) + (e^x - e^{-x})(e^y + e^{-y})$.
8. Expand $(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})$.
9. Solve $2^x = 3$. Solve $0.5^x = 3$. Solve $4^x = 3$.
10. For which values of x (if any) does $1^x = 1$? For which values of x (if any) does $1^x = 3$?
11. For what values of b (if any) does $0^b = 0$? For what values of a (if any) does $a^0 = 0$?

MAT questions

MAT 2015 Q1H
How many distinct solutions does the following equation have?

$$\log_{x^2+2}(4 - 5x^2 - 6x^3) = 2$$

(a) None, (b) 1, (c) 2, (d) 3, (e) Infinitely many.

Hint: that's a scary logarithm! How can we get rid of it?

MAT 2017 Q1I
Let $a, b, c > 0$ and $a \neq 1$. The equation

$$\log_b\left((b^x)^x\right) + \log_a\left(\frac{c^x}{b^x}\right) + \log_a\left(\frac{1}{b}\right)\log_a(c) = 0$$

has a repeated root when

(a) $b^2 = 4ac$, (b) $b = \dfrac{1}{a}$, (c) $c = \dfrac{b}{a}$, (d) $c = \dfrac{1}{b}$, (e) $a = b = c$.

Hint: there's a lot of simplifying to do here before this turns into a polynomial.

MAT 2013 Q1F

Three *positive* numbers a, b, c, satisfy

$$\log_b a = 2, \qquad \log_b (c-3) = 3, \qquad \log_a (c+5) = 2.$$

This information

(a) specifies a uniquely.

(b) is satisfied by exactly two values of a

(c) is satisfied by infinitely many values of a

(d) is contadictory.

Hint: this is secretly a system of equations for a, b, and c. The option "specifies a uniquely" would be true if there is exactly one value of a that works in these equations, and perhaps one or more solutions for b and c. The option "is contradictory" would be true if there are no solutions for at least one of the variables.

MAT 2013 Q1J

For a real number x we denote by $[x]$ the largest integer less than or equal to x.
Let n be a natural number. The integral

$$\int_0^n [2^x] \, \mathrm{d}x$$

equals

(a) $\log_2((2^n-1)!)$, (b) $n2^n - \log_2((2^n)!)$, (c) $n2^n$, (d) $\log_2((2^n)!)$,

where $k! = 1 \times 2 \times 3 \times \cdots \times k$ for a positive integer k.

Hint: split the integral up into different regions where 2^x takes values in between different whole numbers.

Extension

The following material is included for your interest only, and not for MAT preparation.

This isn't on the MAT syllabus, but $\ln x$ plays a special role in calculus. It's the indefinite integral of x^{-1}. Let's explore that. First, here's a quick reminder that we can't integrate x^{-1} with our normal rule for integrating x^n, which would give $\dfrac{x^{n+1}}{n+1}$, because we can't divide by $n+1$ if $n=-1$. But the area under the graph $y=x^{-1}$ from, say, $x=1$ to $x=2$ is just some real number! It's perhaps surprising that it's $\ln 2$.

To get an idea of the link between integrating x^{-1} and $\ln x$, let's write down the problem we're trying to solve;

$$\text{Find a function } y(x) \text{ such that } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x}.$$

Here's a trick – we can flip both sides of this equation the other way up to get

$$\frac{\mathrm{d}x}{\mathrm{d}y}=x$$

This sort of manipulation is definitely not on the MAT syllabus, and it's really not obvious that the inverse of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ should be $\dfrac{\mathrm{d}x}{\mathrm{d}y}$ because the derivative there is not really a fraction, it's more like a notation for a limit, but trust me, this operation does actually work.

Now if we squint at this new equation, it's telling us something about the derivative of x in terms of y (if it helps, switch the x in this equation for something that looks like a fancy y and switch the y for something that looks like a fancy x). It says that when we differentiate x with respect to y, we get x back. That's exactly what e^y does! So we can integrate and write $x=e^y$. Rearranging, this gives $y=\ln x$. Magic!

(Technical note - we could in fact have chosen $x=Ae^y$ for any constant A, and then we'd have got $y=\ln x+c$ for some constant c. This is exactly what we should have expected from the original problem – don't forget the constant of integration!)

Exponentials can also be used to define two new functions;

$$f(x)=\frac{1}{2}\left(e^x+e^{-x}\right) \quad \text{and} \quad g(x)=\frac{1}{2}\left(e^x-e^{-x}\right).$$

These have lots of nice properties which you can check, such as $f'(x)=g(x)$ and $g'(x)=f(x)$ and $f(x)^2-g(x)^2=1$. They are called hyperbolic trigonometric functions, and they're usually written as $f(x)=\cosh x$ and $g(x)=\sinh x$. You can probably guess what $\tanh x$ is.