MAT syllabus

Laws of logarithms and exponentials. Solution of the equation \( a^x = b \).

Revision

- \( a^m a^n = a^{m+n} \) for any positive real number \( a \) and any real numbers \( m \) and \( n \).
- \( (a^m)^n = a^{mn} \) for any positive real number \( a \) and any real numbers \( m \) and \( n \).
- \( a^{-n} = \frac{1}{a^n} \) for any positive real number \( a \) and any real number \( n \).
- \( a^0 = 1 \) for any non-zero real number \( a \).
- The solution to \( a^x = b \) where \( a \) and \( b \) are positive numbers (with \( a \neq 1 \)) is \( \log_a(b) \). In this expression, the number \( a \) is called the base of the logarithm.
- \( \log_a(x) \) is a function of \( x \) which is defined when \( x > 0 \). Like with \( \sin x \), sometimes the brackets are omitted if it’s clear what the function is being applied to, so we might write \( \log_a x \).
- \( \log_a x \) doesn’t repeat any values; if \( \log_a x = \log_a y \) then \( x = y \).
- Note the special case \( \log_a a = 1 \) because \( \log_a a \) is the solution \( x \) to the equation \( a^x = a \), and that solution is 1.
- In fact, \( \log_a(a^x) = x \).
- In that sense, the logarithm function is the inverse function for \( y = a^x \).
- \( a^{\log_a x} = x \).
- \( \log_a(xy) = \log_a(x) + \log_a(y) \).
- \( \log_a(x^k) = k \log_a x \) including \( \log_a \frac{1}{x} = -\log_a x \).
- There’s a mathematical constant called \( e \), which is just a number (it’s about 2.7).
- \( e^x \) is called the exponential function.
- The laws of indices and laws of logarithms above hold when the base \( a \) is equal to \( e \).
- \( \log_e x \) is sometimes written as \( \ln x \) and the function is sometimes called the natural logarithm.
Revision Questions

1. Simplify \((2^3)^4\) and \((2^4)^3\) and \(2^{42}\) and \(2^{32}\).

2. Solve \(x^{-2} + 4x^{-1} + 3 = 0\).

3. Simplify \(\log_{10}3 + \log_{10}4\) into a single term.

4. Write \(\log_3(x^2 + 3x + 2)\) as the sum of two terms, each involving a logarithm.

5. Solve \(\log_x(x^2) = x^3\).

6. Solve \(\log_x(2x) = 3\) for \(x > 0\).

7. Solve \(\log_{x+5}(6x + 22) = 2\).

8. Let \(a = \ln 2\) and \(b = \ln 5\), and write the following in terms of \(a\) and \(b\).

   \[
   \ln 1024, \quad \ln 40, \quad \ln \sqrt[2]{5}, \quad \ln \frac{1}{10}, \quad \ln 1.024.
   \]

9. Expand \((e^x + e^{-x})(e^y - e^{-y}) + (e^x - e^{-x})(e^y + e^{-y})\).

   Expand \((e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})\).

10. Solve \(2^x = 3\). Solve \(0.5^x = 3\). Solve \(4^x = 3\).

11. For which values of \(x\) (if any) does \(1^x = 1\)? For which values of \(x\) (if any) does \(1^x = 3\)?

12. For what values of \(b\) (if any) does \(0^b = 0\)? For what values of \(a\) (if any) does \(a^0 = 0\)?

13. Given \(\log_{10} (\log_{10} x) = 6\), how many zeros are there at the end of the number \(x\)?

14. Solve \(e^x + e^{-x} = 4\).

   How many solutions are there to \(e^x + e^{-x} = c\)? Identify different cases in terms of \(c\).

15. Prove that \(\ln(N + \sqrt{N^2 - 1}) = -\ln(N - \sqrt{N^2 - 1})\) for any number \(N \geq 1\).

16. Consider the equation \(x^y = y^x\) with \(x, y > 0\). Use logarithms to turn this into an equation of the form \(f(x) = f(y)\). [Harder] Sketch \(f(x)\).

17. Simplify \(a^{k\log_a b}\) for positive numbers \(a, b, k\).

18. Consider the number \(x = \log_a b \log_b c\). By simplifying \(a^x\), show that \(x = \log_a c\).

19. Similarly, show that \(\log_a b = \frac{\log_c b}{\log_c a}\) for positive numbers \(a, b, c\), and hence \(\log_a b = \frac{\ln b}{\ln a}\).
MAT questions

MAT 2007 Q1I
Given that \( a \) and \( b \) are positive and
\[ 4 (\log_{10} a)^2 + (\log_{10} b)^2 = 1, \]
then the greatest possible value of \( a \) is

(a) \( 1 \), (b) \( 1 \), (c) \( \sqrt{10} \), (d) \( 10^{\sqrt{2}} \).

MAT 2008 Q1B
Which is the smallest of these values?

(a) \( \log_{10} \pi \), (b) \( \sqrt{\log_{10} (\pi^2)} \), (c) \( \left( \frac{1}{\log_{10} \pi} \right)^3 \), (d) \( \frac{1}{\log_{10} \sqrt{\pi}} \).

MAT 2008 Q1E
The highest power of \( x \) in
\[ \left\{ \left[ (2x^6 + 7)^3 + (3x^8 - 12)^4 \right]^5 \right. \left. + \left[ (3x^5 - 12x^2)^5 + (x^7 + 6)^4 \right]^6 \right\}^3 \]
is

(a) \( x^{424} \), (b) \( x^{450} \), (c) \( x^{500} \), (d) \( x^{504} \).

MAT 2010 Q1E
Which is the largest of the following four numbers?

(a) \( \log_2 3 \), (b) \( \log_4 8 \), (c) \( \log_3 2 \), (d) \( \log_5 10 \).

MAT 2012 Q1C (modified)
Which is the smallest of the following numbers?

(a) \( \left( \sqrt{3} \right)^3 \), (b) \( \log_3 (9^2) \), (c) \( (3 \sin 60^\circ)^2 \), (d) \( \log_2 (\log_2 (8^5)) \).

[See the next page for hints]
Hints

MAT 2007 Q1I

• If I squint at the left-hand side, it looks a bit like the sum of two squares. Let’s write $x = \log_{10} a$ and $y = \log_{10} b$ and see what happens.

MAT 2008 Q1B

• When is $x$ bigger than $\sqrt{2x}$? When is $x$ bigger than $2/x$?

  You’ll need to use the fact that $1 < \pi < 10$, but you shouldn’t need to use any more detailed knowledge of the value of $\pi$ than that.

MAT 2008 Q1E

• What’s the highest power of $x$ in $(2x^6 + 7)^3$? Do not multiply out! Now look at the other terms too.

  We can ignore the outer-most power of 3 while we’re comparing terms, but don’t forget about it at the end.

MAT 2010 Q1E

• You can evaluate one of these exactly. Which one? Next, I would aim to compare the others to that one.

  Here’s a strategy to do that sort of comparison; let’s say that we’re comparing $\log_2 3$ against $\frac{p}{q}$ for some fraction $\frac{p}{q}$. Is $\log_2 3 < \frac{p}{q}$? Well, if it is, then $3 < 2^{p/q}$, so $3^q < 2^p$.

  You’ve got particular values of $p$ and $q$ in mind; go for it!

  You might like to reflect on why it’s OK to manipulate the inequalities like this.

MAT 2012 Q1C

• Simplify each number as much as you can before doing any comparisons.

Extension

[Just for fun, not part of the MAT question]

• Given a positive number $\alpha$, which is the smallest of these values? Identify the different cases according to $\alpha$.

  (a) $\alpha$, (b) $\sqrt{2\alpha}$, (c) $\alpha^{-3}$ (d) $\frac{2}{\alpha}$.

• Which is larger, $(8!)^9$ or $(9!)^8$?