Integrability and complexity in statistical mechanics

Thermodynamic limit vs viscous / dispersive regularisation

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MCNP Mathematics of Complex and Nonlinear Phenomena

Thanks to

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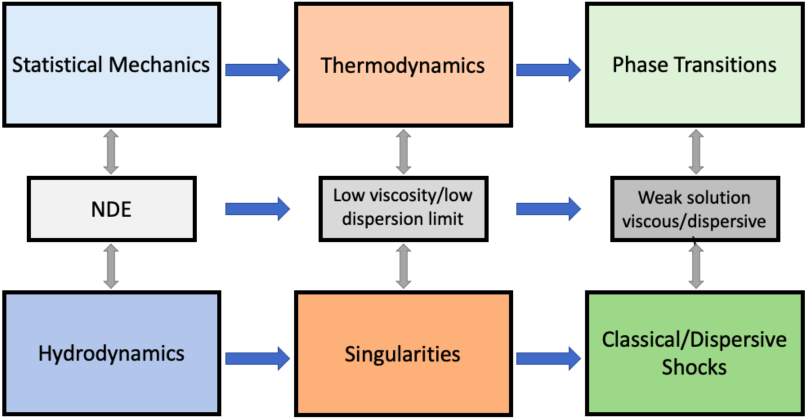
Collaborations

Agliari, Arsie, Barra, Benassi, Biondini, Dell'Atti, De Matteis, Dello Schiavo, De Nittis, Di Lorenzo, Giglio, Guerra, Landolfi, Lorenzoni, Prinari, Senkevich

The big picture



“Case studies”

- Matrix models
  - C. Benassi, M. Dell’Atti, A. Moro (2021) *Symmetric Matrix Ensemble and Integrable Hydrodynamic Chains*, **Letters in Mathematical Physics**, Vol. 111, 78.
  - C. Benassi, A. Moro (2020) *Thermodynamic limit and dispersive regularization in matrix models* **Physical Review E**, Vol. 101, 052118.
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  - P. Lorenzoni, A. Moro (2019) *Exact analysis of phase transitions in mean field Potts models* **Physical Review E**, Vol. 100, 022103.
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- van der Waals type models
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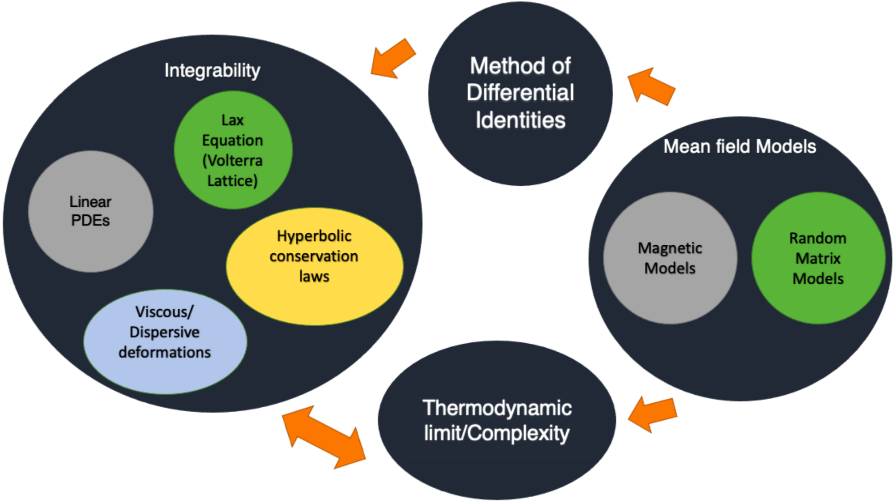
Map of the talk



Method of differential identities

Mean field Potts model

Lorenzoni, M. , PRE 2019

The Hamiltonian is given by

$$H_N = -\frac{J}{2N}\sum_{i,j=1}^{N}\delta(\sigma_i,\sigma_j)$$

where $\delta(\sigma_i,\sigma_j)$ is the Kronecker delta function and $\sigma_i \in \{a_1, a_2, \dots a_q\}$.

Note. The case $q = 2$ corresponds to the mean field Ising model also known as **Curie-Weiss model**.

Let us introduce the *dressed* Hamiltonian

$$H_N^{(d)} = H_N + H_N^{(0)}, \qquad \text{where} \qquad H_N^{(0)} = -\sum_{j=1}^{q} h_j \sum_{i=1}^{N}\sigma_i^j$$

and the partition function

$$Z_N = \sum_{\{\mathcal{C}_N\}} e^{-\beta H_N^{(d)}}$$

where the sum runs over all spin configurations $\mathcal{C}_N$ and $\beta = 1/T$ where T is the temperature.

Key observation:

$$\delta(\sigma_i, \sigma_j) = \sum_{l=1}^{q} \prod_{k \neq l} \frac{\sigma_i - a_k}{a_l - a_k} \frac{\sigma_j - a_k}{a_l - a_k}.$$

Let us consider the case $q = 3$ with $\sigma_i \in \{-1, 0, 1\}$

$$\delta(\sigma_i, \sigma_j) = \frac{3}{2}\sigma_i^2 \sigma_j^2 + \frac{1}{2}\sigma_i \sigma_j - \left(\sigma_i^2 + \sigma_j^2\right) + 1,$$

leading to the Hamiltonian of the form

$$H_N^{(d)} = -\frac{NJ}{2}\left(\frac{1}{2}\mu_1^2 + \frac{3}{2}\mu_2^2 - 2\mu_2\right) - N(h_1\mu_1 + h_2\mu_2),$$

where $\mu_1 = \sum_{i=1}^{N} \sigma_i / N$ and $\mu_2 = \sum_{i=1}^{N} \sigma_i^2 / N$ are the first and second moments.

The differential identity

Introducing the re-scaled variables $t = \beta J/2$, $x = \beta h_1$ and $y = \beta h_2$, we have

$$Z_N = \sum_{\{\mathcal{C}_N\}} e^{N\left[t\left(\frac{1}{2}\mu_1^2+\frac{3}{2}\mu_2^2-2\mu_2\right)+x\mu_1+y\mu_2\right]},$$

We observe that Z_N satisfies the following PDE

$$Z_{N,t} + 2Z_{N,y} = \frac{1}{N}\left(\frac{1}{2}Z_{N,xx} + \frac{3}{2}Z_{N,yy}\right)$$

The associated initial condition is obtained by solving the “free" (linear) model

$$Z_{N,0}(x,y) = Z_N(x,y,0) = (1+2e^y\cosh x)^N$$

for a system of non-interacting spins coupled to the constant external fields x and y.

Free energy

The free energy is defined as

$$F_N = \frac{1}{N}\log Z_N.$$

and satisfies the equation

$$F_{N,t} + 2F_{N,y} = \frac{1}{2}F_{N,x}^2 + \frac{3}{2}F_{N,y}^2 + \frac{1}{N}\left(\frac{1}{2}F_{N,xx} + \frac{3}{2}F_{N,yy}\right)$$

which can be read as a Hamilton-Jacobi type equation with diffusion term of order $O(1/N)$ Away from singularities, when $N \to \infty$

$$F_t + 2F_y - \frac{1}{2}F_x^2 - \frac{3}{2}F_y^2 = 0$$

with initial condition $F(x,y,0) = F_N(x,y,0)$.
For instance, the expectation values of the moments

$$m_{1,N} := \langle \mu_1 \rangle_N = \frac{\partial F_N}{\partial x} \qquad m_{2,N} := \langle \mu_2 \rangle_N = \frac{\partial F_N}{\partial y} \quad \textbf{(Cole-Hopf transform)}$$

Note. $m_{1,N}$ and $m_{2,N}$ satisfy a system of hydrodynamic type.

Thermodynamic limit: solution

Integrating by the method of characteristics, the free energy is given by the formula

$$F = xm_1 + ym_2 + \sum_{k=1}^{3} p_k^2 t - \sum_{k=1}^{3} p_k \log p_k$$

where $m_1 = m_1(x, y, t)$, $m_2 = m_2(x, y, t)$ are stationary points of the free energy

$$x + m_1 t - \frac{1}{2} \log \frac{m_1 + m_2}{m_2 - m_1} = 0$$

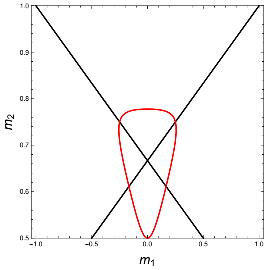
$$y + (3m_2 - 2)t - \frac{1}{2} \log \frac{m_2^2 - m_1^2}{4(m_2 - 1)^2} = 0$$

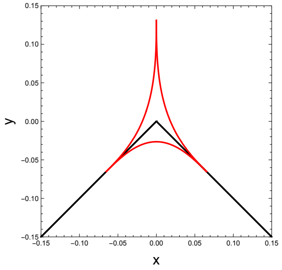
Note:

$$p_1 = \frac{m_1 + m_2}{2} \quad p_2 = \frac{m_2 - m_1}{2} \quad p_3 = 1 - m_2.$$

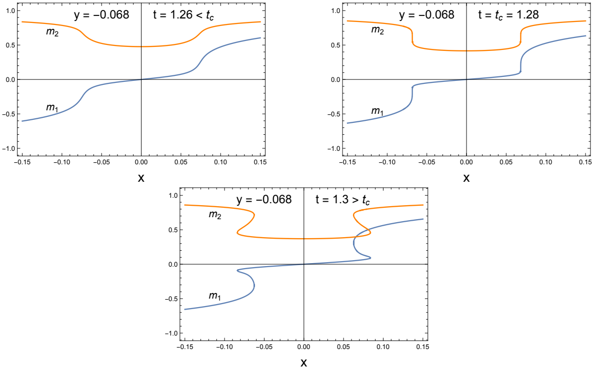
are interpreted as the probabilities of observing macroscopic spin states $+1, -1, 0$.

Critical sector - Whitney cusps locus





Moments



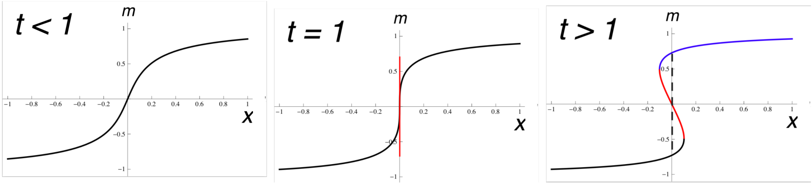
Curie-Weiss case

Curie-Weiss model corresponds to the choice

$$q = 2 \qquad \sigma_i \in \{-1, 1\}$$

We recover the equation of state

$$x + mt = \operatorname{arctanh}(m)$$

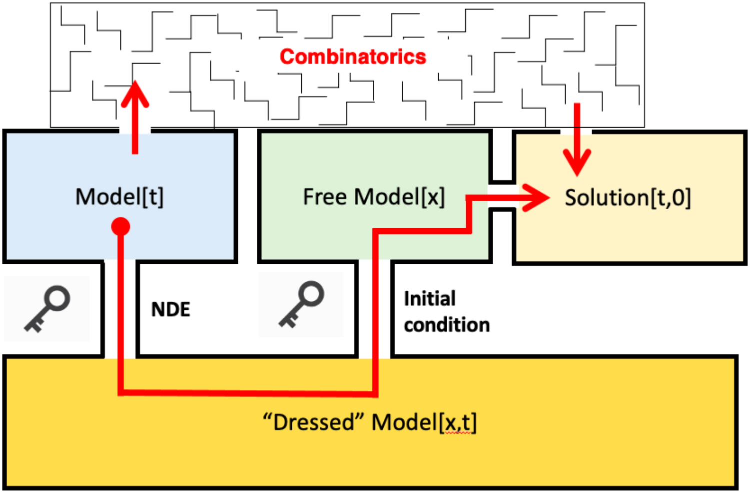


A correspondence table

Thermodynamics		Nonlinear conservation laws
Isothermal/isobaric curves	$\leftrightarrow$	Nonlinear waves
Critical point	$\leftrightarrow$	Gradient catastrophe
Phase transition	$\leftrightarrow$	Shock
Maxwell principle	$\leftrightarrow$	Equal areas rule
Clapeyron Equation	$\leftrightarrow$	Rankine-Hugoniot condition
Triple point	$\leftrightarrow$	Shock confluence
Universality	$\leftrightarrow$	Universality

A.M., Annals of Physics 2014

Method of differential identities: the two-keys



Random Matrix Models

Why Random Matrix Ensembles?

Matrix models are paradigmatic examples of complex systems and arise in a variety of contexts:

- Nuclear Physics
- Statistical Mechanics
- Integrable Systems
- Quantum Field Theory
- Quantum Chaos
- Stochastic processes
- Riemann Zeta function
- Neural Networks/Machine Learning

Extensive list of contributions: Wigner, Dyson, Mehta, Marchenko, Pastur, Tracy, Widom, Witten, Deift, Forrester, McLaughlin, Venakides, Tao, Okounkov, Eynard, Pandharipande, Venakides, Keating, Snaith, Fokas, Its, Brèzin, Itzykson, Parisi, Zuber, Adler, van Moerbeke, Horozov, Eynard, Johansson, Borodin, Grava, Clays, Kuijlaars, Cafasso, Ercolani, Pierce, Jurkiewicz, Ayoama, Kodama, . . . and many others . . .

Hermitian Matrix Models

Let us consider the partition function of the form

$$Z_n(\mathbf{t}) = \int_{\mathcal{H}_n} e^{H(M;\mathbf{t})} dM$$

where

$$H(M) = \mathrm{Tr}\,(-M^2/2 + \sum_{j=1}^{\infty} t_j M^j)$$

is the Hamiltonian function, with $\mathbf{t} = \{t_i\}_{i\geq 1}$ the coupling constants, and the integration is performed over the space $\mathcal{H}_n$ of $n \times n$ Hermitian matrices. Using a classical result (Weyl)

$$Z_n(\mathbf{t}) = \frac{c_n}{n!} \int_{\mathbb{R}^n} \Delta_n^2 \prod_{i=1}^{n} \left(e^{H(\lambda_i;\mathbf{t})}\, d\lambda_i \right),$$

where c_n is constant $\Delta_n = \prod_{1\leq i<j\leq n}(\lambda_i - \lambda_j)$ is the Vandermonde determinant.

Bibliographical note

- Random matrix models and orthogonal polynomials: Dyson $1960's$, Mehta 1960's
- Random matrix models, 2D quantum gravity, integrable systems: Witten 1991; Kontsevich 1992
- Random matrix models and integrable lattices: Gerasimov, Marshakov, Mironov, Morozov, Orlov, 1991; Adler, van Moerbeke 1995
- Integrable systems and genus expansion in TFT: Dubrovin 1992
- Random matrix models and Toda hierarchy (continuum limit): Bonora, Martellini, Xiong 1992; Ercolani, McLaughlin, Pierce 2008
- Chaotic behaviours in Random Matrix models: Jurkiewicz 1991; Senechal 1991
- Dispersive regularization vs continuum limit of Toda lattice: Bloch, Kodama 1991; Deift, McLaughlin 1998
- And many others ...

Hermitian matrix models and Toda Lattice

Let us introduce the function

$$\tau_n(\mathbf{t}) := \frac{Z_n}{c_n} = \frac{1}{n!}\int_{\mathbb{R}^n} \Delta_n^2 \prod_{i=1}^{n}\left(\mathrm{e}^{H(\lambda_i;\mathbf{t})}\, d\lambda_i\right)$$

Consider the tridiagonal symmetric matrix (*Lax matrix*)

$$L = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \dots \\ b_1 & a_2 & b_2 & 0 & \dots \\ 0 & b_2 & a_3 & b_3 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$a_i = \frac{\partial}{\partial t_1}\log\frac{\tau_{i+1}}{\tau_i}, \qquad b_i = \sqrt{\frac{\tau_{i+1}\tau_{i-1}}{\tau_i^2}} \qquad i = 1, 2, \dots$$

Flaschka coordinates: $a_i \to$ moments; $b_i \to$ relative displacements

Theorem (Adler-van Moerbeke)

The function $\tau_n(\mathbf{t})$ is a *tau*-function of the Toda hierarchy, that is L satisfies

$$\frac{\partial L}{\partial t_k} = \left[\frac{1}{2}\left(L^k\right)_s, L\right]$$

where $\left(L^k\right)_s$ denotes the skew-symmetric part of the matrix L^k.

E.g.

$$L_s = \begin{pmatrix} a_1 & b_1 & 0 \\ b_1 & a_2 & b_2 \\ 0 & b_2 & a_3 \end{pmatrix}_s = \begin{pmatrix} 0 & b_1 & 0 \\ -b_1 & 0 & b_2 \\ 0 & -b_2 & 0 \end{pmatrix}$$

Initial condition

Hence, $\tau_n(\mathbf{t})$ is the $\tau-$function for a particular solution of the Toda lattice system fixed by the associated initial conditions

$$a_n(x, \mathbf{0}) = 0 \qquad b_n(\mathbf{0}) = \sqrt{n}.$$

Note 1: The above follows from direct calculation (*Selberg's integral*) of the quantities

$$\frac{\partial}{\partial t_1}\tau_n(\mathbf{t})|_{\mathbf{t}=\mathbf{0}} = 0 \qquad \tau_n(\mathbf{0}) = \frac{(2\pi)^{\frac{n}{2}}}{n!}\prod_{j=1}^{n} j!$$

Note 2: Alternatively, initial conditions can be fixed via Virasoro constraints

The “reduction": Volterra/Kac-van Moerbeke lattice

Benassi, M. , PRE 2020

Let us consider the model with even nonlinear interactions

$$H(M) = \mathrm{Tr}\,(-M^2/2 + \sum_{j=1}^{\infty} t_{2j} M^{2j})$$

which is described by the even flows of the Toda hierachy.
Above initial conditions suggest to look for solutions of the even hierarchy such that

$$a_n(\mathbf{t}) = 0 \qquad n = 1, 2, \ldots$$

leading to the Volterra/Kac-van Moerbeke hierarchy

$$\frac{\partial B_n}{\partial t_{2k}} = B_n(V_{n+1}^{(2k)} - V_{n-1}^{(2k)}).$$

where $B_n = b_n^2$ and $V_n^{(2k)}$ are suitable functions of the variables B_n.

For example, for the first three non-trivial flows we have

$$
\begin{aligned}
V_n^{(2)} &= B_n, \\
V_n^{(4)} &= V_n^{(2)} \left(V_{n-1}^{(2)} + V_n^{(2)} + V_{n+1}^{(2)} \right), \\
V_n^{(6)} &= V_n^{(2)} \left(V_{n-1}^{(2)} V_{n+1}^{(2)} + V_{n-1}^{(4)} + V_n^{(4)} + V_{n+1}^{(4)} \right)
\end{aligned}
$$

Solution

Using the string equation

$$[L, P] = 1$$

where

$$P = -\frac{1}{2} L_s + \sum_{k \geq 1} k t_{2k} \, (L)_s^{2k-1}$$

we obtain the solution of the Volterra system via the recursion relation

$$n = B_n - \sum_{j=1}^{\infty} 2j \, t_{2j} \, V_n^{(2j)}$$

where $V^{(2j)}$ are taken directly from the r.h.s. of

$$\frac{\partial B_n}{\partial t_{2k}} = B_n (V_{n+1}^{(2k)} - V_{n-1}^{(2k)}).$$

This formula generalises to any degree of nonlinearity a result by Brézin, Itzykson, Parisi, Zuber (1978) and Jurkiewicz (1991)

Thermodynamic/Continuum limit

Introduce a positive integer N such that $n/N = O(1)$ and define the interpolating function $B(x)$ such that

$$B(x) = B_n \quad \text{and} \quad B(x \pm \epsilon) = B_{n\pm 1} \quad \text{for} \quad x = n/N$$

with the notation $\epsilon = 1/N$.

Re-scaling

$$u_n = B_n/N = \epsilon B_n \quad T_{2k} = N^{k-1} t_{2k} = t_{2k}/\epsilon^{k-1}$$

with

$$u(x) \quad \text{such that} \quad u(n/N) = u_n$$

Thermodynamic/Continuum limit

Formal Taylor expansion in ϵ

$$u_{T_{2k}} = \sum_{n=0}^{\infty} \epsilon^n g_n^{(k)} \left(u; u_x, \ldots, \partial_x^n u\right)$$

where functions $g_n^{(k)}$ are differential polynomials of u. For instance, the first member of the hierarchy (for $k = 1$), which gives the flow with respect to T_2, takes the following compact form

$$u_{T_2} = 2u \left[\frac{1}{\epsilon} \sinh\left(\epsilon \partial_x\right)\right] u$$

where the operator stays for the formal McLaurin expansion of $\sinh$.

For $\epsilon \to 0$, we get the Hopf equation

$$u_{T_2} = 2uu_x.$$

Similarly, higher flows in T_{2k} lead to higher members of the so-called Hopf hierarchy

$$u_{T_{2k}} = C_k u^k u_x$$

The solution matching the prescribed initial condition follows from the recursion formula leading to (set e.g. $T_{2k} = t_{2k} = 0$ for all $k > 3$)

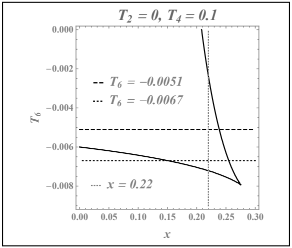
$$x = u - 2T_2 u - 12T_4 u^2 - 60T_6 u^3$$

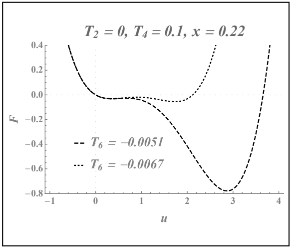
viewed as critical point of the free energy functional of density

$$F[u] = -xu + \left(\frac{1}{2} - T_2\right) u^2 - 4T_4 u^3 - 15T_6 u^4$$

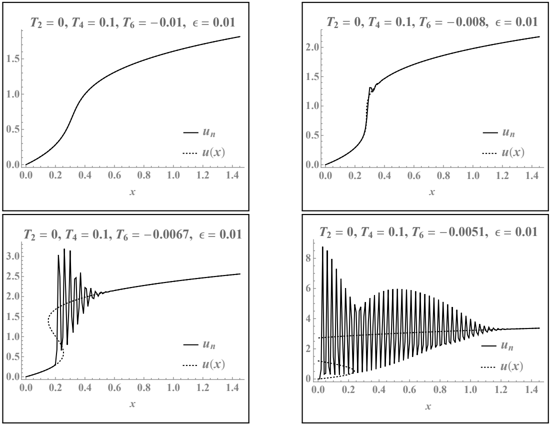
Genus zero phase diagram

$$x = u - 2T_2 u - 12T_4 u^2 - 60T_6 u^3 \qquad F[u] = -xu + \left(\frac{1}{2} - T_2\right) u^2 - 4T_4 u^3 - 15T_6 u^4$$





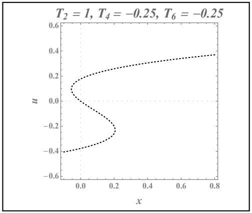
Dispersive shocks vs “chaotic" behaviours

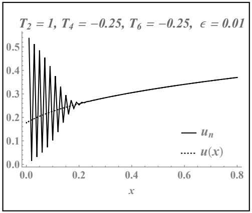


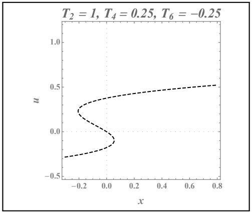
Jurkiewicz, PLB 1991

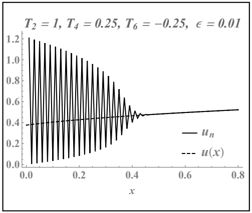
Benassi, M. , PRE 2020

Dispersive shocks vs “chaotic" behaviours

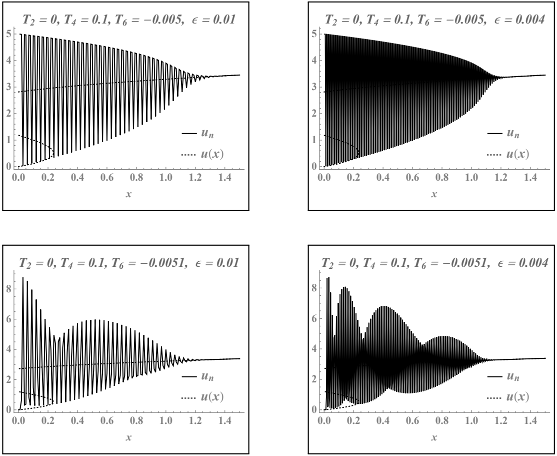








Dispersive shocks vs “chaotic" behaviours



Final remarks

- The method of differential identities provides a general framework for solving a broad class of statistical mechanical models.
- The theory of integrable hydrodynamic systems and their normal forms provide a paradigm for the classification of statistical mechanical models.
- The mathematical connection between Statistical Mechanics, Hydrodynamics and integrability provides an effective approach to the solution of models, understand critical phenomena and may lead to the discovery of new integrable nonlinear differential equations.

Thank you!