

# On singular limits for the Rayleigh-Bénard problem

Eduard Feireisl

based on joint work with P. Bella (TU Dortmund), F. Oschmann (TU Dortmund)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

International PDE conference 2022, University of Oxford  
20 July – 23 July 2022



# Navier–Stokes–Fourier system, primitive system

**Mass conservation:**

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

**Momentum balance:**

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x \rho(\varrho, \vartheta)} = \operatorname{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon} \varrho \nabla_x G}$$

**Internal energy balance (heat equation):**

$$\begin{aligned} \partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) \\ = \boxed{\varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \rho(\varrho, \vartheta) \operatorname{div}_x \mathbf{u} \end{aligned}$$

**Newton's law**

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \mathbb{I}$$

**Fourier's law**

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

## Oberbeck–Boussinesq system, target system

**Incompressibility:**

$$\operatorname{div}_x \mathbf{U} = 0$$

**Momentum balance:**

$$\bar{\varrho} \left( \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \mu(\bar{\vartheta}) \Delta_x \mathbf{U} + r \nabla_x G$$

**Heat equation:**

$$\bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta}) \left( \partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \right) - \bar{\varrho} \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \mathbf{U} \cdot \nabla_x G = \kappa(\bar{\vartheta}) \Delta_x \Theta$$

**Boussinesq relation:**

$$\frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \Theta = \bar{\varrho} \nabla_x G.$$

$$\alpha(\bar{\varrho}, \bar{\vartheta}) \equiv \frac{1}{\bar{\varrho}} \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \left( \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \right)^{-1}$$

$$c_p(\bar{\varrho}, \bar{\vartheta}) \equiv \frac{\partial e(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} + \bar{\varrho}^{-1} \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta}$$

# Spatial domain

## Periodic strip

$$\Omega = \mathcal{T}^{d-1} \times [0, 1]$$
$$\mathcal{T}^{d-1} = ([-1, 1] |_{\{-1;1\}})^{d-1}$$

## Conservative boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbb{S} \cdot \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

## Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \vartheta|_{\partial\Omega} = \vartheta_B$$

## Singular limit in the conservative case

From Navier–Stokes–Fourier to Oberbeck–Boussinesq

$$\begin{aligned}\mathbf{u}_\varepsilon &\rightarrow \mathbf{U} \\ \frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} &\rightarrow r \\ \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} &\rightarrow \Theta\end{aligned}$$

**Convergence for the conservative boundary conditions, EF, A. Novotný 2009**

If the initial data are ill prepared:

$$\mathbf{u}_\varepsilon(0, \cdot), \frac{\varrho_\varepsilon(0, \cdot) - \bar{\varrho}}{\varepsilon}, \frac{\vartheta_\varepsilon(0, \cdot) - \bar{\vartheta}}{\varepsilon} \text{ bounded,}$$

then  $\mathbf{U}$ ,  $r$ , and  $\Theta$  solve the Oberbeck–Boussinesq system with the conservative boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \mathbb{S}(\bar{\vartheta}, \nabla_x \mathbf{U}) \cdot \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0, \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

# Singular limit for the Dirichlet boundary conditions

**Scaled boundary conditions for the temperature:**

$$\vartheta_\varepsilon|_{\partial\Omega} = \bar{\vartheta} + \varepsilon\Theta_B$$

**Initial data:**

$$\mathbf{u}_\varepsilon(0, \cdot) \rightarrow \mathbf{U}_0 \text{ in } L^2(\Omega; R^d), \quad \mathbf{U}_0|_{\partial\Omega} = 0, \quad \operatorname{div}_x \mathbf{U}_0 = 0$$

$$\frac{\varrho_\varepsilon(0, \cdot) - \bar{\varrho}}{\varepsilon} \rightarrow r_0 \text{ in } L^1(\Omega) \text{ and weakly-} (*) \text{ in } L^\infty(\Omega)$$

$$\frac{\vartheta_\varepsilon(0, \cdot) - \bar{\vartheta}}{\varepsilon} \rightarrow \mathfrak{T}_0 \text{ in } L^1(\Omega) \text{ and weakly-} (*) \text{ in } L^\infty(\Omega)$$

**Boussinesq relation**

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r_0 + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathfrak{T}_0 = \bar{\varrho} \nabla_x G.$$

# Convergence for the Dirichlet boundary conditions

Convergence, P. Bella, EF, F. Oschmann 2022:

$$\lambda(\bar{\varrho}, \bar{\vartheta}) = \frac{\bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta})}{\bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta})} \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \in (0, 1)$$

$$\mathbf{u}_\varepsilon \rightarrow \mathbf{U}$$

$$\frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} \rightarrow r$$

$$\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} - \boxed{\lambda(\bar{\varrho}, \bar{\vartheta}) \frac{1}{|\Omega|} \int_\Omega \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} dx} \rightarrow \Theta,$$

where  $(\mathbf{U}, r, \Theta)$  solves the Oberbeck–Boussinesq system with  
*nonlocal boundary conditions*

$$\Theta|_{\partial\Omega} = \Theta_B - \frac{\lambda(\bar{\varrho}, \bar{\vartheta})}{1 - \lambda(\bar{\varrho}, \bar{\vartheta})} \frac{1}{|\Omega|} \int_\Omega \Theta dx$$

# Weak solutions to the primitive system

**Mass conservation:**

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

**Momentum balance:**

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon} \varrho \nabla_x G$$

**Gibbs' relation**

$$\vartheta Ds = De + pD \left( \frac{1}{\varrho} \right)$$

**Entropy balance (inequality):**

$$\begin{aligned} \partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \left( \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) \\ \geq \frac{1}{\vartheta} \left( \varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \end{aligned}$$



## Energy conservation, conservative boundary conditions

**Total energy:**

$$E_\varepsilon(\varrho, \vartheta, \mathbf{u}) = \frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

**Energy balance (inequality):**

$$\begin{aligned} \partial_t E_\varepsilon(\varrho, \vartheta, \mathbf{u}) + \operatorname{div}_x (E_\varepsilon(\varrho, \vartheta, \mathbf{u}) \mathbf{u}) + \operatorname{div}_x (\mathbf{p}(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) \\ = \boxed{(\leq)} \varepsilon^2 \operatorname{div}_x (\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{u}) + \varepsilon \varrho \nabla_x G \cdot \mathbf{u} \end{aligned}$$

**Energy conservation (balance), conservative boundary conditions:**

$$\frac{d}{dt} \int_{\Omega} E_\varepsilon(\varrho, \vartheta, \mathbf{u}) \, dx = (\leq) \varepsilon \int_{\Omega} \varrho \nabla_x G \cdot \mathbf{u} \, dx$$

## Basic properties of weak solutions

- **Global existence.** Under certain physically grounded restrictions imposed on the constitutive relations, the weak solutions exist globally in time for any finite energy initial data
- **Compatibility.** Any sufficiently smooth weak solution is a classical solution of the Navier–Stokes–Fourier system
- **Weak–strong uniqueness.** If  $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$  is a classical solution and  $[\varrho, \vartheta, \mathbf{u}]$  a weak solution starting from the same initial data, then  $\varrho = \tilde{\varrho}$ ,  $\vartheta = \tilde{\vartheta}$ ,  $\mathbf{u} = \tilde{\mathbf{u}}$ .

## Problems with the Dirichlet boundary conditions

Lack of control of the boundary heat flux:

$$\frac{d}{dt} \int_{\Omega} E_{\varepsilon}(\varrho, \vartheta, \mathbf{u}) \, dx + \int_{\partial\Omega} \mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} \, d\sigma_x = (\leq) \varepsilon \int_{\Omega} \varrho \nabla_x G \cdot \mathbf{u} \, dx$$

Ballistic energy:

$$E_{\varepsilon}(\varrho, \vartheta, \mathbf{u}) - \tilde{\vartheta} \varrho s(\varrho, \vartheta) \\ \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B$$

Ballistic energy flux:

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} - \frac{\tilde{\vartheta}}{\vartheta} \mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

## Weak solutions revisited

### Ballistic energy balance

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \varepsilon^2 \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \tilde{\vartheta} \varrho s(\varrho, \vartheta) \, dx \\ & + \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left( \varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx \\ & \leq \int_{\Omega} \left( \varepsilon \varrho \nabla_x G \cdot \mathbf{u} - \varrho s(\varrho, \vartheta) \partial_t \tilde{\vartheta} - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \tilde{\vartheta}}{\vartheta} \right) \, dx \end{aligned}$$

for any  $\tilde{\vartheta} > 0$ ,  $\tilde{\vartheta}|_{\partial\Omega} = \vartheta_B$

- **Compatibility [Chaudhuri–EF 2021].** Smooth weak solutions are classical solutions
- **Weak–strong uniqueness [Chaudhuri–EF 2021].** A weak solution coincides with the strong solution as long as the latter exists

# Relative energy as Bregman distance

Thermodynamic stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho, \vartheta > 0$$

Relative energy:

$$E_\varepsilon(\varrho, \vartheta, \mathbf{u} \mid \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) = \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 \\ + \frac{1}{\varepsilon^2} \left[ \varrho e - \tilde{\vartheta} (\varrho s - \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta})) - \left( e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) + \frac{p(\tilde{\varrho}, \tilde{\vartheta})}{\tilde{\varrho}} \right) (\varrho - \tilde{\varrho}) - \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) \right]$$

Bregmann distance:

$$E_\varepsilon(\varrho, S, \mathbf{m} \mid \tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}) = E_\varepsilon(\varrho, S, \mathbf{m}) \\ - \left\langle \partial_{\varrho, S, \mathbf{m}} E_\varepsilon(\tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}); (\varrho - \tilde{\varrho}, S - \tilde{S}, \mathbf{m} - \tilde{\mathbf{m}}) \right\rangle - E_\varepsilon(\tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}) \\ (\varrho, S, \mathbf{m}) \mapsto E_\varepsilon(\varrho, S, \mathbf{m}) \text{ (strictly) convex}$$

## Relative energy inequality

$$\begin{aligned}
 & \left[ \int_{\Omega} E_{\varepsilon}(\varrho, \vartheta, \mathbf{u} | \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) \, dx \right]_{t=0}^{t=\tau} \\
 & + \int_0^{\tau} \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left( \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta} \right) \, dx dt \\
 \leq & -\frac{1}{\varepsilon^2} \int_0^{\tau} \int_{\Omega} \left( \varrho (s - s(\tilde{\varrho}, \tilde{\vartheta})) \partial_t \tilde{\vartheta} + \varrho (s - s(\tilde{\varrho}, \tilde{\vartheta})) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} - \frac{\kappa(\vartheta) \nabla_x \vartheta}{\vartheta} \cdot \nabla_x \tilde{\vartheta} \right) \, dx dt \\
 & - \int_0^{\tau} \int_{\Omega} \left[ \varrho (\mathbf{u} - \tilde{\mathbf{u}}) \otimes (\mathbf{u} - \tilde{\mathbf{u}}) + \frac{1}{\varepsilon^2} \rho(\varrho, \vartheta) \mathbb{I} - \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \right] : \nabla_x \tilde{\mathbf{u}} \, dx dt \\
 & + \int_0^{\tau} \int_{\Omega} \varrho \left[ \frac{1}{\varepsilon} \nabla_x G - \partial_t \tilde{\mathbf{u}} - (\tilde{\mathbf{u}} \cdot \nabla_x) \tilde{\mathbf{u}} \right] \cdot (\mathbf{u} - \tilde{\mathbf{u}}) \, dx dt \\
 & + \frac{1}{\varepsilon^2} \int_0^{\tau} \int_{\Omega} \left[ \left( 1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t \rho(\tilde{\varrho}, \tilde{\vartheta}) - \frac{\varrho}{\tilde{\varrho}} \mathbf{u} \cdot \nabla_x \rho(\tilde{\varrho}, \tilde{\vartheta}) \right] \, dx dt
 \end{aligned}$$

any trio of continuously differentiable functions  $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$

$$\tilde{\varrho} > 0, \tilde{\vartheta} > 0, \tilde{\vartheta}|_{\partial\Omega} = \tilde{\vartheta}, \tilde{\mathbf{u}}|_{\partial\Omega} = \mathbf{0}.$$

# Constitutive theory, I

## Gibbs' law

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

## Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

## Pressure EOS

$$p(\varrho, \vartheta) = p_m(\varrho, \vartheta) + p_{\text{rad}}(\vartheta)$$

$$p_m(\varrho, \vartheta) = \frac{2}{3}\varrho e_m(\varrho, \vartheta), \quad p_{\text{rad}}(\vartheta) = \frac{a}{3}\vartheta^4, \quad a > 0$$

## Internal energy

$$e(\varrho, \vartheta) = e_m(\varrho, \vartheta) + e_{\text{rad}}(\varrho, \vartheta), \quad e_{\text{rad}}(\varrho, \vartheta) = \frac{a}{\varrho}\vartheta^4.$$

## Constitutive theory, II



$$\text{Gibbs' law} \Rightarrow p_m(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)$$



$$\text{Thermodynamic stability} \Rightarrow \frac{P(Z)}{Z^{\frac{5}{3}}} \searrow p_\infty > 0 \text{ as } Z \rightarrow \infty$$

- **Entropy.**

$$s(\varrho, \vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{4a}{3} \frac{\vartheta^3}{\varrho}, \quad \mathcal{S}' < 0$$

- **Third law of thermodynamics.**

$$\mathcal{S}(Z) \searrow 0 \text{ as } Z \rightarrow \infty$$



## Constitutive theory, III

### Viscosity.

$$0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta), \quad |\mu'(\vartheta)| \leq \bar{\mu}$$

$$0 \leq \eta(\vartheta) \leq \bar{\eta}(1 + \vartheta)$$

### Thermal conductivity.

$$0 < \underline{\kappa}(1 + \vartheta^\beta) \leq \kappa(\vartheta) \leq \bar{\kappa}(1 + \vartheta^\beta), \quad \beta > 6$$

### Global existence [Chaudhuri, EF, 2021].

Under the above constitutive restrictions, the problem admits global-in-time weak solutions for any finite energy initial data and any sufficiently smooth boundary data

## Target problem – smooth solutions

**Incompressibility:**

$$\operatorname{div}_x \mathbf{U} = 0$$

**Momentum balance:**

$$\bar{\varrho} \left( \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \mu(\bar{\vartheta}) \Delta_x \mathbf{U} + r \nabla_x G$$

**Heat equation:**

$$\bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta}) \left( \partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \right) - \bar{\varrho} \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \mathbf{U} \cdot \nabla_x G = \kappa(\bar{\vartheta}) \Delta_x \Theta$$

**Boussinesq relation:**

$$\frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \Theta = \bar{\varrho} \nabla_x G.$$

**Nonlocal boundary conditions**

$$\Theta|_{\partial\Omega} = \Theta_B - \frac{\lambda(\bar{\varrho}, \bar{\vartheta})}{1 - \lambda(\bar{\varrho}, \bar{\vartheta})} \frac{1}{|\Omega|} \int_{\Omega} \Theta \, dx, \quad 0 < \lambda(\bar{\varrho}, \bar{\vartheta}) < 1$$

## Strong solutions of the target problem

**A. Abbatiello, EF , 2022:**

Suppose that

$$G \in W^{1,\infty}(\Omega), \Theta_B \in C^2(\bar{\Omega}),$$

$\Theta_0 \in W^{2,p}(\Omega)$ ,  $\mathbf{U}_0 \in W^{2,p}(\Omega; R^d)$ ,  $\operatorname{div}_x \mathbf{U}_0 = 0$ , for any  $1 \leq p < \infty$ ,  
together with the compatibility conditions

$$\mathbf{U}_0 = 0, \Theta_0 + \frac{\lambda}{1-\lambda} \frac{1}{|\Omega|} \int_{\Omega} \Theta_0 \, dx = \Theta_B \text{ on } \partial\Omega$$

Then there exists  $T_{\max} > 0$ ,  $T_{\max} = \infty$  if  $d = 2$ , such that the target OB system with the initial data

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0, \Theta(0, \cdot) = \Theta_0,$$

admits a (unique) strong solution  $\mathbf{U}$ ,  $\Theta$  in the regularity class

$$\begin{aligned} \mathbf{U} \in L^p(0, T; W^{2,p}(\Omega; R^d)), \partial_t \mathbf{U} \in L^p(0, T; L^p(\Omega; R^d)), \Pi \in L^p(0, T; W^{1,p}(\Omega)), \\ \Theta \in L^p(0, T; W^{2,p}(\Omega)), \partial_t \Theta \in L^p(0, T; L^p(\Omega; R^d)) \end{aligned}$$

for any  $1 \leq p < \infty$  and any  $0 < T < T_{\max}$

# Convergence to the target problem, I

## Well prepared initial data

$$\varrho_\varepsilon(0, \cdot) = \bar{\varrho} + \varepsilon \varrho_{0,\varepsilon}, \quad \bar{\varrho} > 0 \text{ constant}, \quad \int_{\Omega} \varrho_{0,\varepsilon} \, dx = 0$$

$$\vartheta_\varepsilon(0, \cdot) = \bar{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}, \quad \mathbf{u}_\varepsilon(0, \cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\|\varrho_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1, \quad \varrho_{0,\varepsilon} \rightarrow r_0 \text{ in } L^1(\Omega),$$

$$\|\vartheta_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1, \quad \vartheta_{0,\varepsilon} \rightarrow \mathfrak{T}_0 \text{ in } L^1(\Omega),$$

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega; \mathbb{R}^d)} \lesssim 1, \quad \mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{U}_0 \text{ in } L^1(\Omega; \mathbb{R}^d),$$

$\mathfrak{T}_0 \in W^{2,p}(\Omega)$ ,  $\mathbf{U}_0 \in W^{2,p}(\Omega; \mathbb{R}^d)$ , for any  $1 \leq p < \infty$ ,  $\operatorname{div}_x \mathbf{U}_0 = 0$ ,  
 $\mathbf{U}_0 = 0$ ,  $\mathfrak{T}_0 = \Theta_B$  on  $\partial\Omega$ ,

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r_0 + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathfrak{T}_0 = \bar{\varrho} \nabla_x G$$

# Energy estimates

## 1-st ansatz in the relative energy inequality

$$\int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}(0, \cdot), \vartheta_{\varepsilon}(0, \cdot), \mathbf{u}_{\varepsilon}(0, \cdot) \middle| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) dx$$

**Boundedness of initial values:**

$$\int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}(0, \cdot), \vartheta_{\varepsilon}(0, \cdot), \mathbf{u}_{\varepsilon}(0, \cdot) \middle| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) dx \lesssim 1$$

independently of  $\varepsilon \rightarrow 0$

## Convergence to the target problem, II

Convergence as a consequence of energy estimates

$$\frac{\underline{\varrho}_\varepsilon - \bar{\varrho}}{\varepsilon} \rightarrow \mathfrak{R} \text{ weakly-} (*) \text{ in } L^\infty(0, T; L^p(\Omega)), \quad p > 1$$

$$\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} \rightarrow \mathfrak{T} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ and weakly-} (*) \text{ in } L^\infty(0, T; L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d))$$

where

$$\operatorname{div}_x \mathbf{u} = 0$$

$$\mathfrak{T}|_{\partial\Omega} = \Theta_B$$

and

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x \mathfrak{R} + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathfrak{T} = \bar{\varrho} \nabla_x G$$

## Target problem revisited

**New variables:**

$$\mathcal{T}, \quad \mathcal{T} - \lambda(\bar{\varrho}, \bar{\vartheta}) \frac{1}{|\Omega|} \int_{\Omega} \mathcal{T} \, dx = \Theta.$$

**Transformed system:**

$$\operatorname{div}_x \mathbf{U} = 0$$

$$\bar{\varrho} \left( \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \operatorname{div}_x \mathbb{S}(\bar{\vartheta}, \nabla_x \mathbf{U}) + r \nabla_x G$$

$$\begin{aligned} & \bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta}) (\partial_t \mathcal{T} + \mathbf{U} \cdot \nabla_x \mathcal{T}) - \bar{\varrho} \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \mathbf{U} \cdot \nabla_x G \\ & = \kappa(\bar{\vartheta}) \Delta_x \mathcal{T} + \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \partial_t \frac{1}{|\Omega|} \int_{\Omega} \mathcal{T} \, dx \end{aligned}$$

**Boussinesq relation:**

$$\frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathcal{T} = \bar{\varrho} \nabla_x G, \quad \int_{\Omega} r \, dx = 0$$

**Boundary conditions:**

$$\mathbf{U}|_{\partial\Omega} = 0, \quad \mathcal{T}|_{\partial\Omega} = \Theta_B,$$

## Convergence to the target problem, III

**2-nd ansatz in the relative energy inequality**

$$E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \mid \bar{\varrho} + \varepsilon r, \bar{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U} \right)$$

**Convergence of the initial values:**

$$\int_{\Omega} E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \mid \bar{\varrho} + \varepsilon r, \bar{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U} \right) (0, \cdot) \, dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$