

# Dynamics of concentrated vorticities in 2d and 3d Euler flows

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Oxbridge PDE Conference

Oxford, 11 April 2022

## The Euler equation for an incompressible inviscid fluid in $\mathbb{R}^2$ in Vorticity-stream formulation

Find  $(\omega, \psi)$  that solves the **vorticity-stream system**

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-1} \omega & \text{in } \mathbb{R}^2 \times (0, T) \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2 \end{cases} \quad (\text{V})$$

where  $(a, b)^\perp = (b, -a)$ ,

$$(-\Delta)^{-1} f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{1}{|x-y|} f(y) dy.$$

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- The **velocity field** (Biot-Savart law).

$$\mathbf{u}(x, t) = \nabla^\perp \psi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(y-x)^\perp}{|x-y|^2} \omega(y, t) dy$$

- The **vorticity**,  $\omega = \nabla \times \mathbf{u} = \partial_x u_2 - \partial_y u_1$ .

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-1} \omega & \text{in } \mathbb{R}^2 \times (0, T) \\ \omega(\cdot, 0) = \omega_0 & \text{in } \mathbb{R}^2 \end{cases} \quad (V)$$

• **Global existence and uniqueness** in  $L^\infty(\mathbb{R}^2)$  of the initial value problem (V) is known: Wolibner (1933), Yudovich (1963). Solutions are regular if  $\omega_0$  is.

• **Steady states:** if  $\Psi = \Psi(x)$  is a solution of

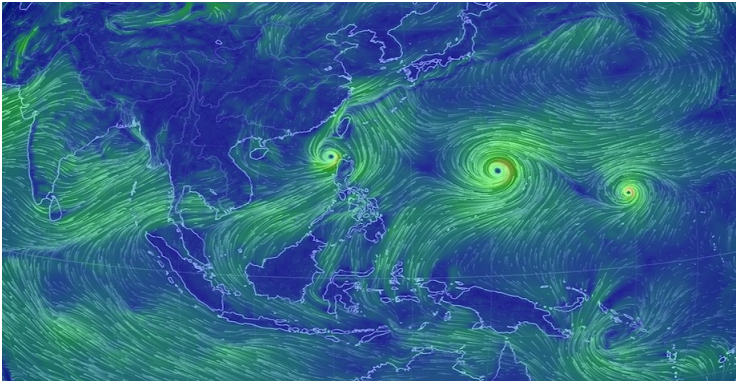
$$-\Delta \Psi = f(\Psi) \quad \text{in } \mathbb{R}^2$$

then  $(\psi, \omega)$ , with  $\omega(x) = f(\psi(x))$ , is a steady state of (V).

• **Example:** Liouville equation:  $-\Delta \psi = e^\psi = \omega = \frac{8}{(1+|x|^2)^2}$ . The Kaufmann-Scully vortex. Mass  $\int_{\mathbb{R}^2} \omega = 8\pi$

We want to describe the evolution of solutions to (V) with vorticities  $\omega(x, t)$  **concentrated** around a finite number of points.

$$\omega(x, t) \approx \sum_{j=1}^k 8\pi\kappa_j \delta(x - \xi_j(t)), \quad \Psi(x, t) \approx \sum_{j=1}^k 4\kappa_j \log \frac{1}{|x - \xi_j(t)|}$$



## Analysis of solutions with highly concentrated vorticities:

A mathematical subject with a long history: it traces back to Helmholtz (1858), Kirchhoff (1876), Routh (1881), Lagally (1921) C.C. Lin (1941).

**Formal  $N$ -vortex singular solutions  $(\omega^s, \Psi^s)$  of  $(\mathbf{V})$ :**

$$\omega^s(x, t) = \sum_{j=1}^N 8\pi\kappa_j \delta(x - \xi_j(t)),$$

where  $\delta(x)$  is the Dirac mass at  $0$ ,  $\kappa_j \in \mathbb{R}$ ,  $\xi_j : [0, T] \rightarrow \Omega$ .  
Since  $\Psi^s = (-\Delta)^{-1}\omega^s$ , we must have

$$\Psi^s(x, t) = \sum_{j=1}^N \kappa_j \Gamma(x - \xi_j(t)), \quad \Gamma(x) = 4 \log \frac{1}{|x|}.$$

Formally we compute

$$\begin{aligned}
& \omega_t^s + \nabla^\perp \Psi^s \cdot \nabla \omega^s \\
&= - \sum_{j=1}^N 8\pi \kappa_j \nabla \delta(x - \xi_j) \cdot \dot{\xi}_j + \sum_{i,j=1}^N 8\pi \kappa_i \kappa_j \nabla^\perp \Gamma(x - \xi_i) \cdot \nabla \delta(x - \xi_j) \\
&= 8\pi \sum_{j=1}^N [-\kappa_j \dot{\xi}_j + \nabla_x^\perp (8\pi \sum_{i \neq j} \kappa_i \kappa_j \Gamma(x - \xi_i))] \cdot \nabla \delta(x - \xi_j).
\end{aligned}$$

We use  $\Gamma(x)$ ,  $\delta(x)$  are “radial”:  $\nabla^\perp \Gamma(x - \xi_j) \cdot \nabla \delta(x - \xi_j) = 0$ .  
 Thus  $(\omega^s, \Psi^s)$  is a “solution” of Problem (V) if and only if  $(\xi_1, \dots, \xi_N)$  solves the planar  $N$ -body problem

$$\boxed{\dot{\xi}_j(t) = \sum_{i \neq j} 4\kappa_i \frac{(\xi_i(t) - \xi_j(t))^\perp}{|\xi_i(t) - \xi_j(t)|^2}, \quad j = 1, \dots, N.} \quad (K)$$

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**A Natural question:** Are there true solutions (smooth) of (V) with vorticities highly concentrated around a finite set of points which evolve by a dynamics approximated by (K)?



We consider  $\omega_{0\varepsilon}$  and  $\Psi_{0\varepsilon}$  explicit  $\varepsilon$ -regularizations of the singular solution

$$\omega^s(x, t) = \sum_{j=1}^N 8\pi\kappa_j \delta(x - \xi_j(t)), \quad \Psi^s(x, t) = \sum_{j=1}^N \kappa_j \log \frac{8}{|x - \xi_j(t)|^4},$$

using Kaufmann-Scully vortices.

$$\omega_{0\varepsilon}(x, t) = \sum_{j=1}^N \frac{\kappa_j}{\varepsilon^2} W_0 \left( \frac{x - \xi_j(t)}{\varepsilon} \right), \quad W_0(y) = \frac{8}{(1 + |y|^2)^2}$$

$$\Psi_{0\varepsilon}(x, t) = \sum_{j=1}^N \kappa_j \log \frac{8}{(|x - \xi_j(t)|^2 + \varepsilon^2)^2}$$

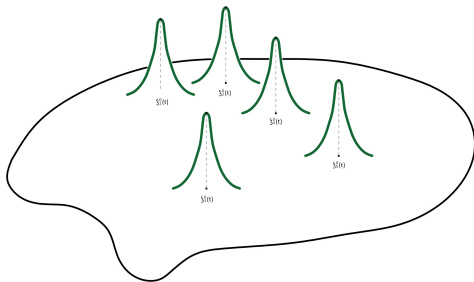
We have  $(-\Delta)^{-1}\omega_{0\varepsilon} = \Psi_{0\varepsilon}$  and  $\int_{\mathbb{R}^2} W_0 = 8\pi$ . We get

$$\omega_{0\varepsilon} \rightarrow \sum_{j=1}^N 8\pi\kappa_j\delta(x-\xi_j(t)), \quad \frac{1}{|\log \varepsilon|}|\nabla^\perp \Psi_{0\varepsilon}|^2 \rightarrow \sum_{j=1}^N 32\pi\kappa_j^2\delta(x-\xi_j(t)).$$

We prove that for a given collisionless solution of the system

$$\dot{\xi}_j(t) = \sum_{i \neq j} \kappa_i 4 \frac{(\xi_i(t) - \xi_j(t))^\perp}{|\xi_i(t) - \xi_j(t)|^2}, \quad j = 1, \dots, N. \quad (K)$$

there is a solution of system (V) that differs little from  $(\Psi_{0\varepsilon}, \omega_{0\varepsilon})$ .



Theorem (Dávila, del Pino, Musso, Wei, ARMA 2020)

Let  $\xi(t)$  be a collisionless solution of (K) in  $[0, T]$ . There exists a solution  $(\omega_\varepsilon, \Psi_\varepsilon)$  of Problem (V) of the form

$$\omega_\varepsilon(x, t) = \sum_{j=1}^k \frac{\kappa_j}{\varepsilon^2} W_0 \left( \frac{x - \xi_j}{\varepsilon} \right) + \phi(x, t)$$
$$\Psi_\varepsilon(x, t) = \sum_{j=1}^k \kappa_j \log \frac{1}{(\varepsilon^2 + |x - \xi_j|^2)^2} + \psi(x, t)$$

where for some  $0 < \sigma < 1$  and all  $(x, t) \in \mathbb{R}^2 \times (0, T)$  we have

$$|\phi(x, t)| \leq \varepsilon^\sigma \sum_{j=1}^k \frac{1}{\varepsilon^2} W_0 \left( \frac{x - \xi_j}{\varepsilon} \right),$$
$$|\psi(x, t)| + \varepsilon |D_x \psi(x, t)| \leq \varepsilon^2.$$

In particular:

$$\omega_\varepsilon \rightarrow \sum_{j=1}^k 8\pi\kappa_j\delta(x - \xi_j) \quad \frac{1}{|\log \varepsilon|} |\nabla \Psi_\varepsilon|^2 \rightarrow \sum_{j=1}^k 8\pi\kappa_j^2\delta(x - \xi_j).$$

A prior result along these lines, Marchioro and Pulvirenti (1993).

### **Ingredients in the construction:**

- Improvement of the approximation in powers of  $\varepsilon$  using elliptic and transport equations.
- Setting up the problem as a coupled system of inner problems near the singularities and an outer problem more regular (the inner-outer gluing scheme)
- A priori estimates to solve by a continuation (degree) argument.

**Improving the approximation.** Let  $\Gamma_0(y) := \log \frac{8}{(1+|y|^2)^2}$ ,

$$\Psi_{0\varepsilon}(x, t) = \sum_{j=1}^k \kappa_j \Gamma_0 \left( \frac{x - \xi_j(t)}{\varepsilon} \right) - \frac{\kappa_j}{8\pi} \log 8\varepsilon^2$$

We want to solve the equation  $E(\omega, \Psi) = 0$ , where

$$E(\omega, \Psi) := \omega_t + \nabla_x^\perp \Psi \cdot \nabla_x \omega, \quad -\Delta_x \Psi = \omega.$$

Near  $\xi_j(t)$  write  $y = \frac{x - \xi_j(t)}{\varepsilon}$ . We look for a solution of the form

$$\Psi = \Psi_{0\varepsilon}(x, t) + \kappa_j \psi(y, t), \quad \omega = \frac{\kappa_j}{\varepsilon^2} W_0(y) + \frac{\kappa_j}{\varepsilon^2} \phi(y, t).$$

In terms of the  $y$ -variable we get the expression

$$\begin{aligned}\varepsilon^4 E(\omega, \Psi) &= \varepsilon^2 \phi_t + (-\varepsilon \dot{\xi} + \nabla_y^\perp \Psi_{0\varepsilon} + \kappa_j \nabla_y^\perp \psi) \cdot \nabla_y (W_0 + \phi), \\ -\Delta_y \psi &= \phi\end{aligned}$$

We have

$$\Psi_{0\varepsilon}(x, t) = \kappa_j \Gamma_0(y) + \varphi(x) + O(\varepsilon^2) + \text{constant}, \quad y = \frac{x - \xi_j}{\varepsilon},$$

$$\varphi(x) = \sum_{i \neq j} \kappa_i \Gamma(x - \xi_i).$$

By assumption  $\dot{\xi}_j = \nabla_x^\perp \varphi(\xi_j)$ , hence we get

$$-\varepsilon \dot{\xi}_j + \nabla_y^\perp \Psi_{0\varepsilon}(\xi_j + \varepsilon y) = \kappa_j \nabla^\perp (\Gamma_0 + \mathcal{R})$$

with  $\mathcal{R} = O(\varepsilon^2 |y|^2)$ .

$$\begin{aligned}\varepsilon^4 E(\omega, \Psi) &= \varepsilon^2 \phi_t + \kappa_j \nabla_y^\perp (\Gamma_0(y) + \mathcal{R} + \psi) \cdot \nabla_y (W_0 + \phi), \\ -\Delta_y \psi &= \phi \\ \mathcal{R} &= O(\varepsilon^2 |y|^2)\end{aligned}$$

Let  $f(u) = e^u$ . Since  $W_0 = f(\Gamma_0)$  we find

$$\begin{aligned}\varepsilon^4 E(\omega, \Psi) &= \varepsilon^2 \phi_t - \kappa_j \nabla_y^\perp \Gamma_0 \cdot \nabla (\Delta \psi + f'(\Gamma_0) \psi) \\ &\quad + \kappa_j \nabla^\perp \mathcal{R} \cdot \nabla U_0 + \kappa_j \nabla^\perp \mathcal{R} \nabla \phi + \nabla^\perp \psi \nabla \phi.\end{aligned}$$

The 0-error term:

$$\varepsilon^4 E(\omega_{0\varepsilon}, \Psi_{0\varepsilon}) = \nabla^\perp \mathcal{R} \cdot \nabla W_0 = O(\varepsilon^2 |y|^{-4}).$$

We obtain a reduction in the error by solving the elliptic equation

$$-\nabla_y^\perp \Gamma_0 \cdot \nabla (\Delta \psi + f'(\Gamma_0) \psi) + \nabla^\perp \mathcal{R} \cdot \nabla W_0 = 0$$

After sufficiently improving the approximation we solve the problem by a continuation (degree) argument, which near each  $\xi_j$  roughly reads as

$$\begin{aligned} \varepsilon^2 \phi_t - \nabla^\perp \Gamma_0 \cdot \nabla (\Delta \psi + f'(\Gamma_0) \psi) + Q(\phi) + E(y, t) &= 0 \\ -\Delta \psi &= \phi \quad \text{in } \mathbb{R}^2 \times [0, T] \end{aligned}$$

with  $E = O(\varepsilon^5 \rho^{-3})$ ,  $Q(\phi) = \nabla^\perp \psi \nabla \phi$ , quadratic term.



**A basic ingredient:** A priori estimates for the linear part of the equation under initial datum zero

$$\begin{cases} \varepsilon^2 \phi_t - \nabla^\perp \Gamma_0 \cdot \nabla (\Delta \psi + f'(\Gamma_0) \psi) + E(y, t) = 0 \\ -\Delta \psi = \phi & \text{in } \mathbb{R}^2 \times [0, T] \\ \phi(\cdot, 0) = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

and  $\int_{B_{\delta/\varepsilon}} y \phi(y, t) dy = 0, \quad \int_{\mathbb{R}^2} \phi(y, t) dy = 0$

$$\|\phi(y, t)(1+|y|^2)\|_{L^2(\mathbb{R}^2)} \lesssim \varepsilon^{-2} |\log \varepsilon|^{1/2} \sup_{\tau \in [0, T]} \|E(y, \tau)(1+|y|^2)\|_{L^2(\mathbb{R}^2)}$$

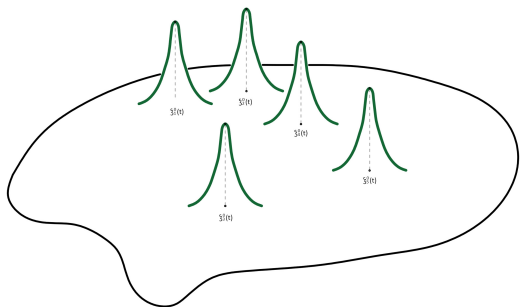
This allows a fixed point scheme to work when  $E = O(\varepsilon^5 \rho^{-3})$ .

# The generalized surface quasigeostrophic equation (SQG)

Let  $\frac{1}{2} < s < 1$ .

$$\begin{cases} \omega_t + \nabla^\perp \psi \cdot \nabla \omega = 0 & \text{in } \mathbb{R}^2 \times (0, T) \\ \psi = (-\Delta)^{-s} \omega & \text{in } \mathbb{R}^2 \times (0, T), \end{cases}$$

$$(-\Delta)^{-s} f(y) = c_s \int_{\mathbb{R}^2} \frac{1}{|x - y|^{2-2s}} f(y) dy$$



$$\dot{\xi}_j(t) = \sum_{i \neq j} \kappa_i d_s \frac{(\xi_i(t) - \xi_j(t))^\perp}{|\xi_i(t) - \xi_j(t)|^{4-2s}}, \quad j = 1, \dots, N. \quad (K_s)$$

Theorem ( MD, Antonio Fernandez )

Let  $0.937 \leq s < 1$ . For a collisionless solution  $\xi(t)$  of the  $N$ -body problem  $(K_s)$  there exists a solution of (SQG) such that

$$\omega(x, t) \approx \sum_{j=1}^N k_j \frac{1}{\varepsilon^2} U_0 \left( \frac{x - \xi_j}{\varepsilon} \right), \quad U_0(y) = \frac{c_s}{(1 + |x|^2)^{1+s}}$$

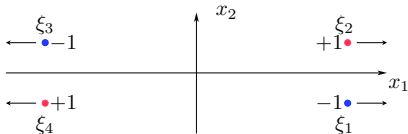
The proof is substantially harder.

**Special case:** A travelling wave solution for the case of a travelling vortex pair: Ao, Dávila, del Pino, Musso, Wei (TAMS, 2021).

## How about $t \rightarrow +\infty$ ?

A 4-vortex situation: (M.D., J. Davila, M. Musso, S. Parmeshwar )

There exists a solution  $w(x, t)$  in Euler  $2d$  which approximately looks like 2 vortex pairs travelling with constant speeds in opposite directions.



$$w(x, t) \approx \varepsilon^{-2} \sum_{j=1}^4 U\left(\frac{x - \xi_j}{\varepsilon}\right) (-1)^j + o(1)$$

$o(1) \rightarrow 0$  as  $t \rightarrow +\infty$

## Nearly singular solutions for Euler in $\mathbb{R}^3$ ?



**Open question:** Solutions with concentrated vorticities near curves (filaments): *the Vortex filament conjecture* (Helmholtz, Da Rios, Levi-Civita 1858-1906-1931).

We consider the Euler equation in  $\mathbb{R}^3$  in stream-vorticity formulation

$$\begin{cases} \omega_t + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} = 0 & , \\ \mathbf{u} = \nabla \times \psi, \quad \psi(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^2} \times \omega(\mathbf{y}, t) d\mathbf{y}. \end{cases} \quad (\text{V})$$

( $\omega = \nabla \times \mathbf{u}$  in  $\mathbb{R}^3$ ). We want to find solutions with vorticity concentrated on a time evolving curve (filament)  $\Gamma(t)$  parametrized by arclength as  $\gamma(s, t)$  in  $\mathbb{R}^3$ .

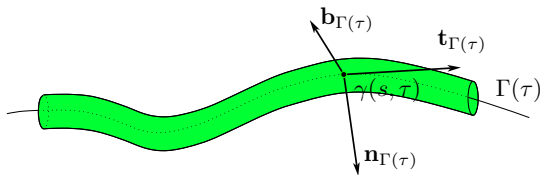
**Vortex filament dynamics.** (After Helmholtz and Kelvin) is a solution  $\omega_\varepsilon(x, t)$  of (V) concentrated in a tube radius  $\varepsilon$  so that

$$\omega_\varepsilon(\cdot, t) \approx c \delta_{\Gamma(t)} \mathbf{t}_{\Gamma(t)} \quad \text{as } \varepsilon \rightarrow 0,$$

$\mathbf{t}_{\Gamma(t)}$  tangent vector field,  $\delta_{\Gamma(t)}$  the uniform curve Dirac measure.  
**1904, Da Rios formal law:** Letting  $\tau = t |\log \varepsilon|$ ,  $\gamma(s, \tau)$  parametrization by arclength of  $\Gamma(\tau)$ ,  $\kappa$  curvature, then

$$\gamma_\tau = \frac{c}{4\pi} (\gamma_s \times \gamma_{ss}) = \frac{c}{4\pi} \kappa \mathbf{b}_{\Gamma(\tau)},$$

$\mathbf{b}_{\Gamma(\tau)}$  binormal vector. This is the *binormal flow of curves*.



The *vortex filament conjecture*:

Let  $\Gamma(\tau)$  be a solution curve of the binormal flow defined in  $[0, T]$  for some  $c > 0$ ,  $T > 0$ . For each  $\varepsilon > 0$  there exists a smooth solution  $\omega_\varepsilon(x, t)$  to (V) satisfying in the distributional sense,

$$\omega_\varepsilon(\cdot, \frac{\tau}{|\log \varepsilon|}) \rightharpoonup c \delta_{\Gamma(\tau)} \mathbf{t}_{\Gamma(\tau)} \quad \text{as } \varepsilon \rightarrow 0, \quad \text{for all } 0 \leq \tau \leq T.$$

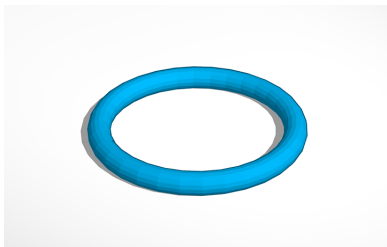
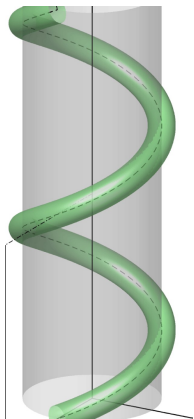
Natural: To look for a solution of the form

$$\omega_\varepsilon(x, \tau) = \frac{1}{\varepsilon^2} W_0 \left( \frac{z}{\varepsilon} \right) \mathbf{t}_{\Gamma(\tau)} + o(1), \quad x = \gamma(\tau, s) + z_1 \mathbf{b}_{\Gamma(\tau)} + z_2 \mathbf{n}_{\Gamma(\tau)},$$

This statement is only known for special curves associated to travelling wave solutions: the thin vortex ring first found by Fraenkel, and recently a helicoidal filament.



**Examples:** a helix whose horizontal section rotates at a constant angular speed or a vertically translating circle are solutions of the bi-normal flow of curves.



Solutions  $\vec{\omega}(x, y, z, t)$  of 3d-Euler with Helicoidal symmetry can be obtained from a scalar function  $w(x + iy, t)$  in the form

$$\vec{\omega}(x, y, z, t) = w(e^{-iz}(x + iy), t/|\log \varepsilon|) \begin{bmatrix} i(x + iy) \\ b \end{bmatrix}$$

where  $w(x, \tau)$  solves

$$\begin{cases} |\log \varepsilon| w_\tau + \nabla^\perp \psi \cdot \nabla w = 0 \\ -\nabla \cdot (K \nabla \psi) = w \end{cases}$$

$$K(x, y) = \frac{1}{\kappa^2 + x^2 + y^2} \begin{pmatrix} \kappa^2 + y^2 & -xy \\ -xy & \kappa^2 + x^2 \end{pmatrix}$$

Rotating helicoidal solutions:

$$w(x + iy, \tau) = w(e^{i\alpha\tau}(x + iy)), \quad \psi((x + iy), \tau) = \psi(e^{i\alpha\tau}(x + iy)).$$

The problem reduces to the elliptic equation

$$-\nabla \cdot (K \nabla \psi) = f(\psi - \frac{\alpha}{2} |\log \varepsilon| (x^2 + y^2)) = w \quad \text{in } \mathbb{R}^2$$

Special case  $f(u) = \varepsilon^2 e^u$ . we prove:

Theorem (Dávila, del Pino, Musso, Wei, CVPDE, to appear)

There exists a solution  $\psi_\varepsilon$  to the equation

$$-\nabla \cdot (K \nabla \psi) = \varepsilon^2 e^{\psi + \lambda(x^2 + y^2)} \quad \text{in } \mathbb{R}^2$$

such that  $\varepsilon^2 e^{\psi - \frac{\alpha}{2} |\log \varepsilon| (x^2 + y^2)} \rightarrow 8\pi \delta_{(x_0, 0)}$ ,  $x_0 > 0$ , for a suitable choice of  $\alpha$ .

$\alpha$  is precisely the number that makes the "rotating helix"

$$\gamma(s, \tau) = \begin{pmatrix} e^{i\left(\frac{s}{\sqrt{b^2 + x_0^2}} - \alpha\tau\right)} (x_0 + iy_0) \\ \frac{bs}{\sqrt{b^2 + x_0^2}} \end{pmatrix}$$

a solution of the binormal flow

Another known solution of the binormal flow that does not change its form in time is the **vortex ring**.

Axisymmetric Euler no-swirl: Cylindrical coordinates

$$\omega(r, z, \tau) = W(r, z, \tau)(-y, x).$$

After rescaling time  $t = \tau/|\log \varepsilon|$ , we get

$$\left\{ \begin{array}{l} |\log \varepsilon| r W_\tau + \nabla^\perp(r^2 \psi) \nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r} \psi_r + \psi_{zz}) := -\Delta_5 \psi = W \\ \psi_r(0, z, \tau) = 0. \end{array} \right.$$

Fraenkel's exact traveling ring solutions (1970-1972):

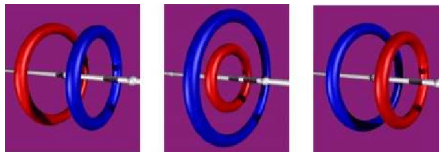
$W(r, z, \tau) = W(r, z - \alpha\tau)$ . It solves the elliptic equation

$$-\alpha r |\log \varepsilon| W_z + \nabla^\perp(r^2 \psi) \cdot \nabla W = 0, \quad -\Delta_5 \psi = W$$

Enough to solve  $-\Delta_5 \psi = F(r^2(\psi - \alpha |\log \varepsilon|))$ .

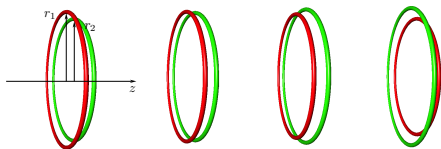


**Interaction of multiple vortex rings:**



**Leapfrogging Vortex-Rings** Helmholtz 1858: predicted the way two identical, coaxial vortex rings interact.

- The rings travel in the same direction. Due to their mutual interaction, the rear ring shrinks and accelerates, and the leading ring widens and decelerates. The rear ring then passes through the leading ring, with this process of *leapfrogging* then repeating again and again.



**Aim:** Mathematically justify the leap-frogging dynamics for the 3d axisymmetric Euler flow without swirl.

$$\begin{cases} |\log \varepsilon| r W_\tau + \nabla^\perp(r^2 \psi) \cdot \nabla W = 0 \\ -(\psi_{rr} + \frac{3}{r} \psi_r + \psi_{zz}) = W \\ \psi_r(0, z, t) = 0 \end{cases}$$

Notation, for a point  $a$  in  $(r, z)$ -plane we write  $a = (r(a), z(a))$ .  
We look for a solution with approximate form

$$W(r, z, \tau) = \sum_{j=1}^2 r(a_j(\tau))^{-1} \frac{1}{\varepsilon_j^2} W_0\left(\frac{x - a_j(\tau)}{\varepsilon_j}\right)$$

Here  $a_j = a_j(\tau)$ ,  $\varepsilon_j(\tau) = \frac{\varepsilon}{\sqrt{r(a_j(\tau))}}$ ,  $W_0(y) = \frac{8}{(1+|y|^2)^2}$

**Theorem** [Dávila, del Pino, Musso, Wei, 2022] Let  $a(t) = (a_1(t), \dots, a_N(t))$  be a collisionless solution of the system

$$\begin{cases} \dot{b}_i(t) = \sum_{j \neq i} \frac{(b_i - b_j)^\perp}{|b_i - b_j|^2} - \frac{r(b_i)}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ b_i(0) = b_i^0 \end{cases}$$

$$a_i(t) = \left( r_0 + \frac{r(b_i(t))}{\sqrt{|\log \varepsilon|}}, z_0 + \frac{t}{r_0} + \frac{z(b_i(t))}{\sqrt{|\log \varepsilon|}} \right)$$

in  $(0, T)$ . Then there exists a solution  $W_\varepsilon$  of 3D axisymmetric Euler flow (without swirl) of the form

$$W_\varepsilon(x, t) = \sum_{j=1}^N \frac{1}{r(a_j)\varepsilon_j^2} W_0 \left( \frac{(r, z) - a_j}{\varepsilon_j} \right) + o(1)$$

$$\varepsilon = \sqrt{r(a_j(t))} \varepsilon_j(t).$$



Thanks for your attention