Propagation of chaos for maxima of particle systems with mean-field drift interaction.

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Outline

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Main idea and standard results

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- $i-$th particle evolution in time:

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where:

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- Initial Conditions: $X_0^{i,N} \sim \nu_0$ (i.i.d)

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- Corresponding McKean-Vlasov SDE:

$$dX_t = C(t, X_{[0,t]})dt + A(t, X_{[0,t]}) \left(B\left(t, X_{[0,t]}, \int g(t, X_{[0,t]}, \boldsymbol{y})\mu_t(d\boldsymbol{y}) \right) dt + dW_t \right)$$

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- Initial Condition: $X_0 \sim \nu_0$
- Propagation of Chaos: for every fixed integer k we have

$$\mathcal{L}\left(X^{1,N}_{[0,T]}, X^{2,N}_{[0,T]}, \ldots, X^{k,N}_{[0,T]}\right) \longrightarrow \mathcal{L}\left(X^{1}_{[0,T]}, X^{2}_{[0,T]}, \ldots, X^{k}_{[0,T]}\right)$$

as $N \longrightarrow +\infty$, where X^1, X^2, ... X^N are i.i.d copies of X.

Main idea and standard results

- Total Variation estimate:

$$\left\| \mathcal{L}\left(X^{1,N}_{[0,T]}, X^{2,N}_{[0,T]}, \dots, X^{k,N}_{[0,T]}\right) - \mathcal{L}\left(X^{1}_{[0,T]}, X^{2}_{[0,T]}, \dots, X^{k}_{[0,T]}\right) \right\|_{TV,(0,T)} \leq C(T)\sqrt{\frac{k}{N}}$$

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- In a Gaussian setting we can also bound

$$KL\left(\mathcal{L}\left(X^{1,N}_{[0,T]}, X^{2,N}_{[0,T]}, \ldots, X^{k,N}_{[0,T]}\right), \mathcal{L}\left(X^{1}_{[0,T]}, X^{2}_{[0,T]}, \ldots, X^{k}_{[0,T]}\right)\right) \geq c(T)\left(\frac{k}{N}\right)^2$$

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$$\geq c(T)\left(\frac{k}{N}\right)^2$$

- For $k = N$: RHS of the last $= \mathcal{O}(1) \Rightarrow$ convergence of the whole system does **NOT** hold in general

Outline

Functions allowing propagation of chaos

- A function F_N of N arguments is said to allow propagation of chaos if

$$\lim_{N\to\infty} F_N\left(X_t^{1,N}, X_t^{2,N}, ..., X_t^{N,N}\right) = \lim_{N\to\infty} F_N\left(X_t^{1}, X_t^{2}, ..., X_t^{N}\right)$$

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 in distribution.
- Clearly, functions F_N depending nicely only on the first k particles (k fixed positive integer) allow propagation of chaos
- Propagation of chaos strong estimates involve the relative entropies of the two systems $\Rightarrow$ an F_N preserving little information of the system of its arguments is more likely to allow propagation of chaos

Functions allowing propagation of chaos

- In many cases, symmetric averages of the form:

$$F_N\left(X_t^{1,N}, X_t^{2,N}, \ldots, X_t^{N,N}\right) = \frac{1}{N}\sum_{i=1}^{N} g\left(X_t^{i,N}\right) = \int g(p_t\boldsymbol{y})d\mu_t^N(\boldsymbol{y})$$

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can be shown to allow propagation of chaos, where p_t is the projection mapping a path defined on $[0, t]$ to its value at $t > 0$.

- Interesting question: given a function F_N of the whole system $\left\{X_t^{1,N}, X_t^{2,N}, \ldots, X_t^{N,N}\right\}$, investigate whether it allows propagation of chaos

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$$F_N\left(X_t^{1,N}, X_t^{2,N}, ..., X_t^{N,N}\right) = \max_{i \leq N} \frac{X_t^{i,N} - b_t^N}{a_t^N} = \frac{\max_{i \leq N} X_t^{i,N} - b_t^N}{a_t^N}$$

allow propagation of chaos or not, where $b_\cdot^N$ and $a_\cdot^N$ are deterministic path-valued sequences such that the weak limit of the above as $N \to \infty$ is non-trivial for any $t \geq 0$.

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- Why is that important?
- Answer: For an infinite i.i.d sequence $X^1, X^2, \ldots$ of random variables, there is a complete theory for the asymptotic behaviour of normalized maxima of the form

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  as $N \to \infty$ (extreme value theory).
- Therefore, if normalized maxima allow propagation of chaos, we can also evaluate weak non-trivial limits of

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- Extreme value theory tells us that weak non-trivial limits belong to a family of Gumbel, Weibull and Frechet distributions
- Conclusion: If normalized maxima allow propagation of chaos, we can extend extreme value theory to particle systems with mean-field drift interaction

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- Particle systems with mean-field drift interaction are used to describe the values or the default intensities of the assets in large portfolios
- The study of the top performing or the worst performing assets leads to the study of the maxima of the particle systems

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Example (Gaussian system with mean-reversion towards the average)

For $A(t, x_{[0,t]}) = \sigma$, $B(t, x_{[0,t]}, r) = -\kappa(x_t - r)/\sigma$, $C(t, x_{[0,t]}) = 0$ and $g(t, x_{[0,t]}, y_{[0,t]}) = y_t$, we obtain the system

$$X_t^{i,N} = X_0^i - \kappa \int_0^t \left(X_s^{i,N} - \frac{1}{N} \sum_{j=1}^{N} X_s^{j,N} \right) ds + \sigma W_t^i, \quad i = 1, \dots, N,$$

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- Used for modelling:
  - monetary reserves of banks
  - default intensities in large interbank networks.

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- McKean-Vlasov SDE solution X: Gaussian Ornstein-Uhlenbeck process with constant mean m_0 and time-t variance given by

$$\sigma_t^2 = \mathrm{Var}(X_t) = e^{-2\kappa t}\sigma_0^2 + (1 - e^{-2\kappa t})\frac{\sigma^2}{2\kappa}.$$

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- Normalizing sequences:

$$b_t^N = m_0 + \sigma_t \sqrt{2\log N - \log\log N - \log(4\pi)}$$
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- Normalized maximum limiting distribution: Standard Gumbel with CDF: $F(x) = \exp(-e^{-x})$

Second example

Example (Rank-based particle system)

For $A(t, x_{[0,t]}) = \sqrt{2}$, $B(t, x_{[0,t]}, r) = B(r)$, $C(t, x_{[0,t]}) = 0$ and $g(t, x_{[0,t]}, y_{[0,t]}) = \mathbf{1}_{\{y_t \leq x_t\}}$, we obtain the system

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- In the rank-based system, the drift of the i-th particle equals $B(k_t/N)$, where k_t counts the number of particles having smaller values at any $t > 0$.
- Rank-based systems play an important role in stochastic portfolio theory

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\begin{aligned}
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- Normalizing sequences: determined via standard extreme value theory techniques*
- Normalized maximum limiting distribution: Standard Gumbel with CDF: $F(x) = \exp(-e^{-x})$**
- *,**: Only when X_t is stationary and B satisfies certain conditions. Otherwise, the question is still open

Outline

Assumptions

Assumption 1

The coefficient functions satisfy the following conditions:

- A and C are uniformly bounded,
- for every $t > 0$ and every continuous path $\boldsymbol{x} = x_{[0,t]}$ defined on $[0,\ t]$, the function $r \mapsto B(t, \boldsymbol{x}_{[0,t]}, r)$ is twice continuously differentiable, and its first and second derivatives are bounded uniformly in $(t, \boldsymbol{x})$.

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- Note that $r \mapsto B(t, \boldsymbol{x}_{[0,t]}, r)$ itself need not be bounded, only its first two derivatives. This applies to the first example we covered, where $B(t, \boldsymbol{x}_{[0,t]}, r)$ had uniformly linear growth in r.

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- If the interaction function g is uniformly bounded, the growth properties of $r \mapsto B(t, \boldsymbol{x}_{[0,t]}, r)$ become irrelevant. This applies to the first example we covered, where g is just an indicator

Assumptions

Assumption 2 (Moment bounds)

Assume that there is a continuous function $K(t)$ such that for all $p \in \mathbb{N}$, $t > 0$, $N \in \mathbb{N}$, and $i, j \in \{1, \dots, N\}$, one has the moment bounds

$$\mathbb{E}\left[g(t, X^i_{[0,t]}, X^j_{[0,t]})^{2p}\right] \leq p!\, K(t)^p \tag{1}$$

and

$$\mathbb{E}\left[\left(\int g(t, X^i_{[0,t]}, \boldsymbol{y})(\mu^N_t - \mu_t)(d\boldsymbol{y})\right)^{2p}\right] \leq \frac{1}{N^p} p!\, K(t)^p. \tag{2}$$

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- Assumption 2 holds when $g(t, X^i_{[0,t]}, X^j_{[0,t]}) - \int g(t, X^i_{[0,t]}, \boldsymbol{y})\mu_t(d\boldsymbol{y})$ are bounded or conditionally (on $X^i_{[0,t]}$) sub-Gaussian with a uniformly bounded variance proxy. This covers a very wide range of mean-field systems (including both of our examples).

Outline

Changing the probability measure

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- Define the stochastic exponential $Z^N = \exp(M^N - \frac{1}{2}\langle M^N \rangle)$, where

$$M_t^N = \sum_{i=1}^{N} \int_0^t \Delta B_s^{i,N} dW_s^i$$

and

$$\Delta B_t^{i,N} = B\left(t, X_{[0,t]}^i, \int g(t, X_{[0,t]}^i, \boldsymbol{y}) \mu_t^N(d\boldsymbol{y})\right) - B\left(t, X_{[0,t]}^i, \int g(t, X_{[0,t]}^i, \boldsymbol{y}) \mu_t(d\boldsymbol{y})\right).$$

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- Under our assumptions, we can show that Z^N is a true Martingale
- Girsanov's theorem: For a probability measure Q^N with $\frac{d\mathbb{Q}^{\mathbb{N}}}{d\mathbb{P}} = Z_T^N$, we have that

$$\mathcal{L}_{Q^N}\left(X_{[0,T]}^1, X_{[0,T]}^2, \dots, X_{[0,T]}^k\right) = \mathcal{L}_P\left(X_{[0,T]}^{1,N}, X_{[0,T]}^{2,N}, \dots, X_{[0,T]}^{k,N}\right)$$

The main result

Theorem

Suppose Assumptions 1 and 2 are satisfied and consider the laws $\mathbb{Q}^N$ constructed above. Fix $t \in (0, t)$ and suppose that for some normalizing constants a_t^N, b_t^N the normalized maxima of the i.i.d. system converge weakly to a nondegenerate distribution function Γ_t on $\mathbb{R}$:

$$\mathbb{P}\left(\max_{i \leq N} \frac{X_t^i - b_t^N}{a_t^N} \leq x\right) \to \Gamma_t(x) \text{ as } N \to \infty, \quad x \in \mathbb{R}.$$

Then the normalized maxima of the interacting particle systems also converge to Γ_t:

$$\mathbb{Q}^N\left(\max_{i \leq N} \frac{X_t^i - b_t^N}{a_t^N} \leq x\right) \to \Gamma_t(x) \text{ as } N \to \infty, \quad x \in \mathbb{R}.$$

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