Geometry

MAT syllabus

Co-ordinate geometry and vectors in the plane. The equations of straight lines and circles. Basic properties of circles. Lengths of arcs of circles.

Revision

- Points in the plane can be described with two co-ordinates \((x, y)\). The \(x\)-axis is the line \(y = 0\), and the \(y\)-axis is the line \(x = 0\).

- A vector \(\begin{pmatrix} x \\ y \end{pmatrix}\) can store the same information as a pair of co-ordinates. Used in that sense, the vector is called a position vector.

- A vector can also describe the displacement from one point to another, so that \(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\) could represent the displacement from \((1, 1)\) to \((3, 2)\) for example.

- Vectors can be added by adding the components separately. To show that in a diagram, we might interpret the first vector as a position vector (drawing an arrow starting from the origin) and then interpret the second as a displacement (drawing an arrow starting from the end of the first vector).

- The magnitude of the vector \(\begin{pmatrix} x \\ y \end{pmatrix}\) is \(\sqrt{x^2 + y^2}\).

- The distance from \(A\) to \(B\) is the magnitude of the vector displacement from \(A\) to \(B\). The distance from \((x_1, y_1)\) to \((x_2, y_2)\) is \(\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\).

- A vector can be multiplied by a number by multiplying each component by that number. The result is a vector in the same direction but with scaled magnitude.

- A straight line has equation \(y = mx + c\), where \(m\) is the gradient and \(c\) is the \(y\)-intercept. Other ways to write the equation of a line are \(ax + by + c = 0\) (where that’s a different \(c\) to the one in the previous expression) or \(y - y_1 = m(x - x_1)\). The last expression is useful because that line goes through the point \((x_1, y_1)\) and has gradient \(m\), which might be information that we’ve been given.

- Two lines are parallel if and only if they have the same gradient. Two lines are perpendicular if and only if their gradients multiply to give \(-1\).

- The equation of the circle with centre \((a, b)\) and radius \(r\) is \((x - a)^2 + (y - b)^2 = r^2\).

- The angle in a semicircle is a right angle; if \(AB\) is the diameter of a circle, and \(C\) is on the circle, then \(\angle ACB = 90^\circ\).

For solutions see: [www.maths.ox.ac.uk/r/matlive](http://www.maths.ox.ac.uk/r/matlive)
• The tangent is at right angles to the radius at any point on a circle’s circumference.

• A circle with radius \( r \) has area \( \pi r^2 \) and circumference \( 2\pi r \).

• If two radii of a circle of radius \( r \) make an angle of \( \theta < 180^\circ \) (in degrees), then the length of the minor arc between those radii is \( \frac{\theta}{360^\circ}2\pi r \). The area of the sector enclosed by that arc and the radii is \( \frac{\theta}{360^\circ}\pi r^2 \).

Revision Questions

1. Draw a diagram to show the three separate position vectors \( \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) and \( \begin{pmatrix} -4 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -2 \end{pmatrix} \).

2. Add the vectors \( \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) and \( \begin{pmatrix} -4 \\ 1 \end{pmatrix} \). Show this on your diagram.

3. Find \( 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \). Show this on your diagram.

4. Find the equation of the line through the points (1, 5) and (3, -1).

5. Find the equation of the line through the point (3, 5) with gradient 2.

6. Show that the points (0, 5), (1, 3), (2, 6), and (3, 4) lie on the corners of a square.

7. Find equations of three lines such that the finite region bounded by the three lines is an equilateral triangle.

8. A circle has centre \((-1, 2)\) and radius 3. Write down an equation for the circle. What’s the area of this circle? Where does this circle cross the axes?

9. A circle is given by \( x^2 + 9x + y^2 - 3y = 10 \). Find the centre and radius of the circle.

10. Points \( A \) and \( B \) lie on a circle with centre \( O \) and radius 2. The angle \( \angle AOB \) is \( 120^\circ \). Find the length of the arc between \( A \) and \( B \). Find the area enclosed by that arc and the radii \( OA \) and \( OB \).

11. Two circles are given by \( x^2 + y^2 = 4 \) and \( (x - 2)^2 + y^2 = 4 \). Find the area of the region that’s inside both circles.

12. The points \((0, 0)\) and \((1, a)\) and \((0, a + a^{-1})\) all lie on the same circle. Find the centre of the circle in terms of \( a \).

13. A circle has centre \((c, 0)\) and radius 1. The area in the region \( x > 0 \) which is inside the circle depends on \( c \), and we’ll call it \( A(c) \). Sketch a graph of \( A(c) \) against \( c \).
MAT Questions

MAT 2016 Q1C
The origin lies inside the circle with equation

\[ x^2 + ax + y^2 + by = c \]

precisely when

(a) \( c > 0 \), (b) \( a^2 + b^2 > c \), (c) \( a^2 + b^2 < c \), (d) \( a^2 + b^2 > 4c \), (e) \( a^2 + b^2 < 4c \).

[See the next page for hints]

MAT 2017 Q1G
For all \( \theta \) in the range \( 0 \leq \theta < 360^\circ \) the line

\[ (y - 1) \cos \theta = (x + 1) \sin \theta \]

divides the disc \( x^2 + y^2 \leq 4 \) into two regions. Let \( A(\theta) \) denote the area of the larger region. Then \( A(\theta) \) achieves its maximum value at

(a) one value of \( \theta \), (b) two values of \( \theta \), (c) three values of \( \theta \), (d) four values of \( \theta \), (e) all values of \( \theta \).

[See the next page for hints]

MAT 2014 Q1D
The reflection of the point \( (1,0) \) in the line \( y = mx \) has coordinates

(a) \( \left( \frac{m^2 + 1}{m^2 - 1}, \frac{m}{m^2 - 1} \right) \), (b) \( (1,m) \), (c) \( (1-m,m) \), (d) \( \left( \frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2} \right) \), (e) \( (1-m^2,m) \).

[See the next page for hints]
Hints

MAT 2016 Q1C

- The circle $x^2 + ax + y^2 + by = c$ is written in quite an unusual way. Where is the centre of this circle? What’s the radius?
- Given the location of the centre and the radius, how can you check whether another point is inside the circle?
- If you use the Pythagorean theorem for a distance, be careful with inequalities.

MAT 2017 Q1G

- The line $(y - 1) \cos \theta = (x + 1) \sin \theta$ is written in quite an unusual way. Can you find any points that lie on the line?
- Are there any points $(x, y)$ that lie on the line for all values of $\theta$?
- What happens as $\theta$ changes? Sketch some special cases as $\theta$ changes from 0 to 360°.
- $A(\theta)$ is the area of the “larger” region. Is this region always on the same side of the line?

MAT 2014 Q1D

- The line segment between $(1, 0)$ and the reflection of that point should meet the line $y = mx$ at right angles, and the midpoint of the line segment should lie on $y = mx$.
- Two lines are at right angles if their gradients multiply to $-1$.
- If you draw a diagram, you might spot a pair of congruent triangles. Or you could set up some algebra to express the idea that the midpoint lies on the line $y = mx$. 

For solutions see [www.maths.ox.ac.uk/r/matlive](http://www.maths.ox.ac.uk/r/matlive)
MAT 2008 Q4

Let $p$ and $q$ be positive real numbers. Let $P$ denote the point $(p,0)$ and let $Q$ denote the point $(0,q)$.

(i) Show that the equation of the circle $C$ which passes through $P$, $Q$, and the origin $O$ is

$$x^2 - px + y^2 - qy = 0.$$ 

Find the centre and area of $C$.

(ii) Show that

$$\frac{{\text{area of circle } C}}{{\text{area of triangle } OPQ}} \geq \pi$$

(iii) Find expressions for the angles $OPQ$ and $OQP$ if

$$\frac{{\text{area of circle } C}}{{\text{area of triangle } OPQ}} = 2\pi$$

[See the next page for hints]
Hints

(i) My strategy for this part is to solve in the opposite order; I’ll write down an equation for the circle which I’m happy with, and then I’ll make it look like the one in the question. In general the equation for a circle is \((x - a)^2 + (y - b)^2 = r^2\). We can plug in the points that we know lie on the circle and we’ll get equations for \(a\) and \(b\) and \(r\).

To find the centre and radius, I can either rearrange the expression in the question to make it look more like a familiar equation for a circle. Or I could use some of my working above.

Alternatively, draw a right-angle on your copy of the diagram and recall a geometric fact about circles.

(ii) We know expressions for both of these areas in terms of \(p\) and \(q\). Then we’ve got an inequality to prove.

To prove an inequality like \(a^2 + b^2 \geq 2ab\), we might move all the terms to one side and try to spot a square (because squares are positive or zero).

(iii) The inequality is now an equality! Time to solve for \(p\) and \(q\)... except we can’t. Not entirely, that is. There’s a bit of ambiguity because if we double \(p\) and also double \(q\) then both areas will go up by a factor of 4, so the equality will still hold. At best, we can work out the ratio between \(p\) and \(q\). This is enough to find the angles in terms of inverse trigonometric functions (can you see why?)

There’s another ambiguity; we don’t know which way round \(P\) and \(Q\) are (which one is larger? We don’t know). Hopefully, we’ll get an equation with at least two solutions.

Extension

[Just for fun, not part of the MAT question]

- Here’s an equation for \(\tan 2\theta\) (not on the MAT syllabus).

\[
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}
\]

Set \(\theta\) to be the value of angle \(OPQ\) you found in this question, for which you know the value(s) of \(\tan \theta\). Calculate \(\tan 2\theta\). Deduce the exact value(s) of \(\theta\) in degrees.

- Find the minimum value of \(x + x^{-1}\) for \(x > 0\).
The three corners of a triangle $T$ are $(0,0)$, $(3,0)$, $(1,2h)$ where $h > 0$. The circle $C$ has equation $x^2 + y^2 = 4$. The angle of the triangle at the origin is denoted as $\theta$. The circle and triangle are drawn in the diagrams above for different values of $h$.

(i) Express $\tan \theta$ in terms of $h$.

(ii) Show that the point $(1, 2h)$ lies inside $C$ when $h < \sqrt{3}/2$.

(iii) Find the equation of the line connecting $(3,0)$ and $(1, 2h)$.
    Show that this line is tangential to the circle $C$ when $h = 2/\sqrt{5}$.

(iv) Suppose now that $h > 2/\sqrt{5}$. Find the area of the region inside both $C$ and $T$ in terms of $\theta$.

(v) Now let $h = 6/7$. Show that the point $(8/5, 6/5)$ lies on both the line (from part (iii)) and the circle $C$.

Hence show that the area of the region inside both $C$ and $T$ equals

$$\frac{27}{35} + \frac{\alpha \pi}{90^\circ}$$

where $\alpha$ is an angle in degrees whose tangent, $\tan \alpha$, you should determine.

[You may use the fact that the area of a triangle with corners at $(0,0)$, $(a,b)$, $(c,d)$ equals $\frac{1}{2} |ad - bc|$.

[See the next page for hints]
Hints

(i) Drop a perpendicular line from (1, 2h) to the x-axis. Your equation for tan θ should be nice and simple.

(ii) Points inside the circle have $x^2 + y^2 < 4$, because the distance from such a point to the origin is less than 2. For this question, this is an inequality involving h. Try to rearrange it for h.

(iii) Careful; the point of tangency is not (1, 2h). To be tangential, we would need a single point which is on the line and also on the circle $x^2 + y^2 = 4$.

(iv) In order to get our picture right, we’ll need to know whether that point is inside or outside the circle. A good way to compare numbers like $\sqrt{a/b}$ and $c/\sqrt{d}$ is to compare their squares.

Remember that we know the area of a sector of a circle.

(v) This case is different from the previous part. Again, check whether the point (1, 2h) lies inside the circle using part (ii). We’ll need to compare some square roots again.

The question gives us the coordinates of a point where the line crosses the circle. Mark this on your diagram.

The part of the answer that’s a rational number could come from the area of a triangle using the hint at the end of the question. The part of the answer that involves angles and $\pi$ is related to the area of a circle. We might guess that an expression like this comes from the area of a triangle plus the area of a sector.

Extension

[Just for fun, not part of the MAT question]

- If you’ve learned about the area of a trapezium, drop perpendiculars from $(a, b)$ and $(c, d)$ to the x-axis, identify the areas of two right-angled triangles and one trapezium, and deduce the fact about triangle area given in this question.

- If you’ve learned about the vector product (also known as the cross product), explain the fact about triangle area given in this question.