MAT syllabus

Derivative of x^a , including for fractional exponents. Derivative of e^{kx}. Derivative of a sum of functions. Tangents and normals to graphs. Turning points. Second order derivatives. Maxima and minima. Increasing and decreasing functions. Differentiation from first principles. Indefinite integration as the reverse of differentiation. Definite integrals and the signed areas they represent. Integration of x^a (where $a \neq -1$) and sums thereof.

Revision

- The derivative of x^a is ax^{a-1}, including for fractional exponents like $a = \frac{1}{2}$.
- If k is a constant then he derivative of e^{kx} is ke^{kx}.
- If a is a constant then the derivative of $af(x)$ is a times the derivative of $f(x)$.
- The derivative of $y_1 + y_2$ is (the derivative of y_1)+(the derivative of y_2). Perhaps this looks too obvious to need stating, but remember that (from a different bit of maths), the square of $y_1 + y_2$ is not equal to (the square of y_1) plus (the square of y_2).
- The tangent to a graph at a particular point is a line which has the same value and derivative as the graph at that point. So if we want the tangent to the graph $y = x^2$ at $x = 3$, we need the value of y (which is 9), and the value of the derivative (which is $2x = 6$). So the tangent line has derivative 6 and value 9. The derivative of a line is its gradient, so we can write $y = 6x + c$ and solve for c using the value at $x = 3$ to get $y = 6x - 18$.
- The normal to a graph is a line which has the same value and is at right angles to the tangent. Two lines are at right angles if their gradients multiply to -1. So at the point above, we would want $y = (-1/6)x + c$ and, since the line goes through $(3, 9)$, we have $c = 19/2$.
- If the derivative changes sign $(+/-)$ at a point, that's a turning point. You'll have zero derivative at the turning point, but that's not actually sufficient for the derivative to *change* sign (e.g. x^3 has zero derivative at $x = 0$, but that's not a turning point because the derivative is positive on both sides).
- The derivative of a derivative is called the second derivative. You can work out the derivatives one at a time. So the second derivative of x^a would be the derivative of ax^{a-1}, which is $a(a - 1)x^{a-2}$. In general, the second derivative of e^{kx} is k^2e^{kx}. The second derivative is the rate of change of the derivative.
- "Maxima" is the plural of "maximum". "Minima" is the plural of "minimum". A turning point is a local maximum if the second derivative is negative at that point,

or it's a local minimum if the second derivative is positive. The word "local" here means that very near to that point, the function takes its maximum value. Overall, the function might have several local maxima, and it might increase without bound (like $y = x$ for example!) so just having second derivative zero might not mean that that's biggest value of the function. Over an interval like $[0, 1]$ the function might take its maximum at a local maximum, or maybe at one of the endpoints (like how $y = x$ would take its maximum value at $x = 1$ over that interval).

- If the derivative is positive, that's an increasing function. If it's negative, that's a decreasing function. In general a function might increase in some regions and decrease in other regions.

- If you have two points on a graph, you can join the line between them – that's called the chord. If instead you find the tangent at one of the points, that gives you the gradient of the graph at that point. If you move the second point closer and closer to the first point, then the gradient of the chord gets closer and closer to the gradient of the tangent, which is the value of the derivative at that point. Calculating the gradient of the chord is a nice and sensible thing to do; it's just $\frac{y_2 - y_1}{x_2 - x_1}$, so this is called a "first principles" approach to differentiation.

- Indefinite integration (without limits) is the reverse of differentiation in the sense that if the derivative of $f(x)$ is $g(x)$ then the indefinite integral of $g(x)$ is $f(x) + c$ where c could be any constant. You can use this to integrate anything that you could have got out of differentiating.

- The integral of x^n is $\frac{x^{n+1}}{n+1}$, provided that $n \neq 1$.

- A definite integral is written like $\int_a^b f(x)\,\mathrm{d}x$ where a and b are the two end-points, and it is the difference in value of the indefinite integral at the two end-points; $F(b) - F(a)$ where the derivative of $F(x)$ is $f(x)$.

- If $f(x) > 0$ for $a < x < b$ then $\int_a^b f(x)\,\mathrm{d}x$ is the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

- If $f(x) < 0$ for $a < x < b$ then $\int_a^b f(x)\,\mathrm{d}x$ is minus one times the area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$. Areas are supposed to be positive. The integral here is sometimes called the "signed area" to reflect the fact that it's got a minus sign.

- If $f(x)$ is sometimes positive and sometimes negative in $a < x < b$ then split into separate regions where $f(x)$ is positive or negative before applying the above.

- $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$

Warm-up

- Differentiate $x^{17} - x^{-17}$ with respect to x.
- Differentiate $2\sqrt{x} + 3\sqrt[3]{x}$ with respect to x.
- Differentiate $1 - e^{3x}$ with respect to x.
- Find the tangent to the curve $y = e^x + x^2$ at $x = 2$.
- Find the normal to the parabola $y = x^2$ at $x = 3$.
- Find the turning points of the curve $y = x^4 - 2x^3 + x^2$. Identify whether the turning points are maxima or minima. For which values of x is $y = x^4 - 2x^3 + x^2$ increasing? For which values of x is it decreasing?
- Two points A and B are on the curve $y = x^3 + x^2 + x + 1$. A is held fixed at $(1, 4)$. The point B is moved along the curve towards A. What happens to the line through A and B?
- Suppose that the derivative of a polynomial $p(x)$ with respect to x is $q(x)$. Find an expression for $\int q(x)\,\mathrm{d}x$.
- Find the area enclosed between $x = -1$ and $x = 1$ by the polynomial $y = x^4 - 4x^2 + 3$ and the x-axis.
- Find $\int_{-1}^{1} 1 + x + x^2 + x^3 + x^4 + x^5 + x^6\,\mathrm{d}x$.
- Find $\int_{-\pi}^{\pi} x^{2021}\,\mathrm{d}x$.
- Find
$$\int \frac{x+3}{x^3}\,\mathrm{d}x, \qquad \int \sqrt[3]{x}\,\mathrm{d}x, \qquad \int \left(\left(x^2\right)^3\right)^5\,\mathrm{d}x, \qquad \int \left(x^2+1\right)^3\,\mathrm{d}x$$
- By thinking about the area that the integral represents, explain why
$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{1} f(-x)\,\mathrm{d}x.$$

MAT questions

MAT 2014 Q1C

The cubic

$$y = kx^3 - (k+1)x^2 + (2-k)x - k$$

has a turning point, that is a local minimum, when $x = 1$ precisely for

(a) $k > 0$, (b) $0 < k < 1$, (c) $k > \frac{1}{2}$, (d) $k < 3$, (e) all values of k.

Hint: we'll need to check the second derivative.

MAT 2017 Q1A

Let

$$f(x) = 2x^3 - kx^2 + 2x - k.$$

For what values of the real number k does the graph $y = f(x)$ have two distinct real stationary points?

(a) $-2\sqrt{3} < k < 2\sqrt{3}$

(b) $k < -2\sqrt{3}$ or $2\sqrt{3} < k$

(c) $k < -\sqrt{21} - 3$ or $\sqrt{21} - 3 < k$

(d) $-\sqrt{21} - 3 < k < \sqrt{21} - 3$

(e) all values of k.

Hint: The derivative of $f(x)$ is a quadratic. You know lots about quadratics!

MAT 2014 Q1J

For all real numbers x, the function $f(x)$ satisfies

$$6 + f(x) = 2f(-x) + 3x^2 \left(\int_{-1}^{1} f(t)\,\mathrm{d}t \right).$$

It follows that $\int_{-1}^{1} f(x)\,\mathrm{d}x$ equals

(a) 4, (b) 6, (c) 11, (d) $\frac{27}{2}$, (e) 23.

Hint: $\int_{-1}^{1} f(x)\,\mathrm{d}x$ is just a number, and it's equal to $\int_{-1}^{1} f(t)\,\mathrm{d}t$ and equal to $\int_{-1}^{1} f(-x)\,\mathrm{d}x$.

MAT 2016 Q1H

Consider two functions

$$\begin{aligned} f(x) &= a - x^2 \\ g(x) &= x^4 - a. \end{aligned}$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

(a) all values of a, (b) $a > 1$, (c) $a > \frac{6}{5}$,

(d) $a > \left(\frac{4}{3}\right)^{\frac{3}{2}}$, (e) $a > \left(\frac{6}{5}\right)^{4}$.

Hint: there are lots of words in this question, but the thing we need to do is pretty standard! First find the points where $f = 0$ and calculate that area in terms of a. Do the same for g.

MAT 2017 Q3

For each positive integer k, let $f_k(x) = x^{1/k}$ for $x \geqslant 0$.

(i) On the same axes, labelling each curve clearly, sketch $y = f_k(x)$ for $k = 1, 2, 3$ indicating the intersection points.

(ii) Between the two points in (i), the curves $y = f_k(x)$ enclose several regions. What is the area of the region between the graphs of $y = f_k(x)$ and $y = f_{k+1}(x)$? Verify that the area of the region between f_1 and f_2 is $\frac{1}{6}$.

Let c be a constant where $0 < c < 1$.

(iii) Find the x-coordinates of the points of intersection of the line $y = c$ with $y = f_1(x)$ and of $y = c$ with $y = f_2(x)$.

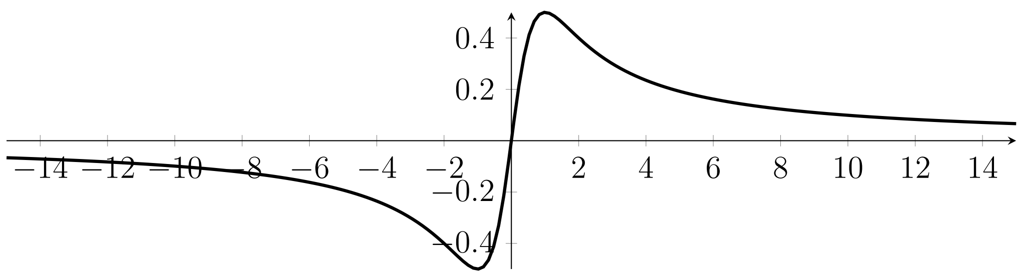
(iv) The constant c is chosen so that the line $y = c$ divides the region between $y = f_1(x)$ and $y = f_2(x)$ into two regions of equal area. Show that c satisfies the cubic equation $4c^3 - 6c^2 + 1 = 0$. Hence find c.

Hints: Draw a really large sketch. Most of the question is about $f_1(x) = x$ and $f_2(x) = \sqrt{x}$, so if you don't like the notation $f_1(x)$, you can replace that everywhere with just x.

Extension

The following material is included for your interest only, and not for MAT preparation.

Here's a sketch of $y = f(x)$ for the function $f(x) = \dfrac{x}{1+x^2}$.



This function decays to zero for very large positive x, or for very negative x, but the area under the curve between $x = 0$ and $x = b$ is unbounded for large b; the actual value is

$$\int_0^b \frac{x}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\log(1+b^2)$$

which gets really big for large b. So $\int_0^\infty f(x)\,\mathrm{d}x$ doesn't exist for this function.

What if we want $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ instead?

Here's the problem; there's more than one thing that might mean!

One possible definition is $\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_0^\infty f(x)\,\mathrm{d}x + \int_{-\infty}^0 f(x)\,\mathrm{d}x$, where the first integral is the limit of $\int_0^b f(x)\,\mathrm{d}x$ for large b and the second is the limit of $\int_{-a}^0 f(x)\,\mathrm{d}x$ for large a.

Another possible definition is $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ is the limit of $\int_{-b}^b f(x)\,\mathrm{d}x$ for large b.

Another possible definition is $\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ is the limit of $\int_{-b}^{2b} f(x)\,\mathrm{d}x$ for large b.

And there are other possible definitions.

Sometimes it doesn't matter which one you mean. But for $f(x) = \dfrac{x}{1+x^2}$, the first interpretation is problematic, because neither integral converges; there's an infinitely large area under the curve for positive x and an infinitely large negative contribution from negative x, and we can't work those out separately. The second interpretation does give a number; it's zero. The third one also gives a number, but it's $\ln 2$; a different answer!

In statistics, the probability distribution function $\dfrac{1}{\pi(1+x^2)}$ is called the Cauchy distribution. Its mean and variance are undefined. If you take random samples from the Cauchy distribution, then they tend to be clustered around zero, but as you take more samples, the sample mean doesn't settle down to zero. Occasionally, but not very often, you get a large number out of the Cauchy distribution which brings the sample mean away from zero. That's not what you might expect to happen!