How do degenerate mobilities determine singularity formation in Cahn-Hilliard equations?

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Joint work with Andreas Münch and Amy Novick-Cohen

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Oxford Mathematics

- Spinodal decomposition.
- Coarsening.

Polymer mixture at ratio 70/30. *Cabral, Higgins, Yerina, Magonov 2002*

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Phase-field models:

- Smooth phase field variable $u \approx \pm 1$ away from interface.
- u transitions between $+1$ and -1 across interface region of width $O(\varepsilon)$, $\varepsilon \ll 1$.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a bounded and convex domain, $\partial\Omega \in C^{1,1}$, $T > 0$ and $0 < \varepsilon \ll 1$, $u = u(x,t)$,

$$
\begin{aligned}
\partial_t u &= -\nabla \cdot \mathbf{j}, \quad \text{in} \quad \Omega \times (0,T), \\
\mathbf{j} &= -M(u)\nabla \mu, \\
\mu &= -\varepsilon^2 \Delta u + f'(u),
\end{aligned}
$$

where $\mathbf{j}$ is the flux, $M \geq 0$ the mobility, μ the chemical potential and f the homogeneous free energy.

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where $\mathbf{j}$ is the flux, $M \geq 0$ the mobility, μ the chemical potential and f the homogeneous free energy. With boundary and initial conditions:

$$\begin{aligned} \nabla u \cdot \mathbf{n} &= 0, \quad \text{on } \partial\Omega \times (0,T), \quad \textit{(Neumann)} \\ \mathbf{j} \cdot \mathbf{n} &= 0, \quad \text{on } \partial\Omega \times (0,T), \quad \textit{(no flux)} \\ u(\cdot,0) &= u_0 \quad \text{on } \Omega. \end{aligned}$$

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Conservation of mass $m(t) := \int_\Omega u(x,t)dx$.

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Conservation of mass $\mathrm{m}(t) := \int_\Omega u(x,t)\,dx$.

Decaying energy $E[u](t) := \int_\Omega \left[\frac{\varepsilon^2}{2}|\nabla u|^2 + f(u)\right] dx$.

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- Polymer blends (e.g. De Gennes 1980, Castellano & Glotzer 1995)
- Surface diffusion and electromigration in crystals and alloys (e.g. Cahn, Elliott & Novick-Cohen 1996; Barrett, Garcke & Nürnberg 2007; Dziwnik, Münch, Wagner 2017)

Choice of f and M in $\partial_t u = \nabla\cdot(M(u)\nabla(-\varepsilon^2\Delta u + f'(u)))$

Double well free energy:

$$f(u) := \frac{(1-u^2)^2}{2}$$

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$$M_0(u) := 1;$$

$$M_n(u) := (1-u^2)_+^n;$$

$n \in \mathbb{R}^+$. Note $M_n(\pm 1) = 0$.

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Elliott and Garcke 1996: Let $T > 0$ and $u_0 \in H^1(\Omega)$ with $|u_0| \leq 1$ plus assumptions on entropy of initial data. Then there exists a weak solution $|u| \leq 1$ in $\Omega \times (0; T)$.

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Are there $n > 0$ that ensure $|u| < 1$? What happens when $|u| \to 1$?

Cahn-Hilliard equation in one dimension, $\Omega \subset \mathbb{R}$,

$$\partial_t u = \partial_x [M(u)(-\varepsilon^2 \partial_{xxx} u + \partial_x f'(u))].$$

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$$\partial_t h = -\varepsilon^2 2^n \partial_x [h^n \partial_{xxx} h],$$

which models **thin liquid films driven by surface tension.**

Cahn-Hilliard equation in one dimension, $\Omega \subset \mathbb{R}$,

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which models thin liquid films driven by surface tension. We couple it with

$$\partial_x h(x,t) = 0, \quad \partial_{xxx} h(x,t) = 0, \; x \in \partial\Omega \qquad \text{(Neumann)};$$

$$h(x,t) = 1, \quad \partial_{xx} h(x,t) = p, \quad p \in \mathbb{R}, \; x \in \partial\Omega \qquad \text{(Fixed Pressure)},$$

$$h(\cdot, 0) = h_0, \quad \text{on } \Omega .$$

Cahn-Hilliard equation in one dimension, ` R,

$$\partial_t u \quad \partial_x \; M(u)(\quad \varepsilon^2 \partial_{xxx} u \quad \partial_x f'(u)) \;:$$

Take $M \quad M_n(u) \quad (1 \quad u^2)^n$ and $h \quad 1 \quad u \, C \, 0$. If hSP 1 then the highest order terms are

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which models thin liquid lms driven by surface tension. We couple it with

$$\partial_x h(x;) \quad 0; \quad \partial_{xxx} h(x;) \quad 0; \; x > \partial \qquad \text{(Neumann)};$$

$$h(x;) \quad 1; \quad \partial_{xx} h(x;) \quad p; \quad p > R; \; x > \partial \qquad \text{(Fixed Pressure)}$$

$$h(\; ;0) \quad h_0; \qquad \text{on} \quad ;$$

Are there n A 0 that ensure h A 0?

Cahn-Hilliard equation in one dimension, $\Omega \subset \mathbb{R}$,

$$\partial_t u = -\partial_x\left(M(u)\left(\varepsilon^2\partial_{xxx}u - \partial_x f'(u)\right)\right).$$

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Are there $n > 0$ that ensure $h > 0$? What happens if $h = 0$?

Constantin, Elgindi, Nguyen, Vicol 2018: $n \; 1$. Pressure b.c. with $p \; A \; 2$. The solution must pinch o in either nite or in nite time, i.e.

$$\inf_{1;1 \quad 0;T\bullet} h \quad 0;$$

for some $T > ˆ 0; ª$. Any solution that touches 0 in nite time becomes singular.

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Bernis and Friedman 1990: ˆ 1; 1•. Neumann b.c. h_0 C 0 plus assumptions on entropy of initial data.

- If 1 @ n @ 2, then h C 0.
- If 2 B n @ 4, then h C 0 and ˜ h 0• has zero measure.
- If n C 4, then h A 0 and the solution is unique.

Solution $u(x,t)$ is expected to converge to a stationary solution $U(x)$

$$-\varepsilon^2 \Delta U - f'(U) = \mu_c, \quad \mu_c \in \mathbb{R}.$$

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Niethammer 1995: Existence and uniqueness (up to $U \mapsto -U$) of small energy stationary solutions.

U

r

Back to Cahn-Hilliard: Stationary 2D radial case with $M_0(u) = 1$

Solution $u(x;t)$ is expected to converge to a stationary solution $U(x)$

$$\varepsilon^2 \Delta U = f'(U) - \mu_c, \quad \mu_c \in \mathbb{R}.$$

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$\varepsilon|\log(\varepsilon)|$

$O(\varepsilon)$

U

$O(\varepsilon)$

r

Lee, Münch, Süli 2016; Pesce, Münch 2021: For $u_0 \equiv 1$, numerical solution u develops a maximum less but close to 1 near interface, where $M_2(u) \approx 0$.

S

r‡

Time dependent 2D radial case with $M_2(u) = (1 - u^2)^2$

Lee, Münch, Süli 2016; Pesce, Münch 2021: For S, u_0 SB 1, numerical solution u develops a maximum less but close to 1 near interface, where $M_2(u) \approx 0$.

S

r_*

Does touchdown happen in finite or infinite time?

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S

$r_‡$

Does touchdown happen in finite or infinite time? Does it depend on n?

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S

$r_\ddagger$

Does touchdown happen in finite or infinite time? Does it depend on n? Does it have some underlying structure?

Time dependent 2D radial case in $B = B_1(0)$

Let $u = u(r,t)$,

$$\partial_t u = \frac{1}{r}\partial_r\big(r M(u)\partial_r \mu\big),$$

$$\mu = -\varepsilon^2 \frac{1}{r}\partial_r(r\,\partial_r u) + f'(u),$$

for $(r,t) \in (0,1)\times(0,\infty)$, under boundary conditions

$$\partial_r u(1,t) = 0, \quad M(u(1,t))\,\partial_r \mu(1,t) = 0,$$

$$\partial_r u(0,t) = 0, \quad \partial_r \mu(0,t) = 0,$$

$$u(r,0) = u_0(r),$$

and where

$$f(u) = \frac{(1-u^2)^2}{2}, \quad M(u) = (1-u^2)^n, \quad n > 0.$$

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and where

$$f(u) = \frac{(1-u^2)^2}{2}; \quad M(u) = (1-u^2)^n; \quad n \geq 0.$$

Consider the Lebesgue and Sobolev spaces of radial functions $L^2_{rad}(B)$, $H^p_{rad}(B)$.

Theorem (Novick-Cohen and Pesce 2022+)

Let $u_0 \in H^1_{rad}(B)$ with $|u_0| \leq 1$ plus assumptions on entropy of initial data.
Then $\exists u \in L^2(0,T;H^2_{rad}(B)) \cap L^\infty(0,T;H^1_{rad}(B)) \cap C(0,T;L^2_{rad}(B))$ such that $|u| \leq 1$

Theorem (Novick-Cohen and Pesce 2022+)

Let u_0 >H^1_{rad}ˆB• with §u_0§B 1 plus assumptions on entropy of initial data. Then §u >L^{2}ˆ 0; T ;H^2_{rad}ˆB•• 9 L^{a} ˆ0; T ; H^1_{rad}ˆB•• 9 Cˆ 0; T ; L^2_{rad}ˆB•• such that §u§B 1 and

S$_0^T$ ` ˆt•; @$_t$uˆt•e$_{H^1;H^{1}}$ dt S$_0^T$ S$_0^1$ j@ rdrdt;

S$_0^T$ S$_0^1$ j rdrdt "^{2} S$_0^T$ S$_0^1$ $\frac{1}{r}$@ˆr @u• @ˆMˆu• • rdrdt
S$_0^T$ S$_0^1$ ˆMf^{œœ}• u•@$_t$u rdrdt;

for all >L^{2}ˆ0; T ; H^1_{rad}ˆB•• and >L^{2}ˆ0; T ; H^1_{rad}ˆB•• 9 L^{a} ˆB_T• such that $\frac{}{r}$ >L^{2}ˆ0; T ; L^2_{rad}ˆB•• which satisfy 0 on ˆ0; T • ˜0; 1•.

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$$S_0^T \; `\;ˆt•; @_t uˆt•e_{H^1;H^{1}} dt \qquad S_0^T S_0^1 j@_t \; rdrdt;$$

$$S_0^T S_0^1 j \quad rdrdt \qquad "^2 S_0^T S_0^1 \frac{1}{r} @ˆr@u• @ˆMˆu• \; • rdrdt$$

$$S_0^T S_0^1 ˆMf^{œœ}•u•@_t u \quad rdrdt;$$

for all >L^2ˆ0; T ; H^1_{rad}ˆB•• and >L^2ˆ0; T ; H^1_{rad}ˆB•• 9 $L^{ª}$ ˆB_T• such that $\frac{}{r}$ >L^2ˆ0; T ; L^2_{rad}ˆB•• which satisfy 0 on ˆ0; T • ˜0; 1•.

- Based on proof by Elliott and Garcke 1996.

Theorem (Novick-Cohen and Pesce 2022+)

Let $u_0 \in H^1_{rad}(B)$ with $|u_0| \le 1$ plus assumptions on entropy of initial data. Then $\exists u \in L^2(0,T;H^2_{rad}(B)) \cap L^\infty(0,T;H^1_{rad}(B)) \cap C(0,T;L^2_{rad}(B))$ such that $|u| \le 1$ and

$$\int_0^T \langle \xi(t), \partial_t u(t)\rangle_{H^1,H^{-1}}\, dt - \int_0^T \int_0^1 j\, \partial_r \xi \, r\,dr\,dt;$$

$$\int_0^T \int_0^1 j \, r\,dr\,dt = \varepsilon^2 \int_0^T \int_0^1 \frac{1}{r}\partial_r(r\,\partial_r u)\,\partial_r(M(u)\,\cdot)\, r\,dr\,dt - \int_0^T \int_0^1 (Mf''(u))\,\partial_r u \, r\,dr\,dt;$$

for all $\xi \in L^2(0,T;H^1_{rad}(B))$ and $\zeta \in L^2(0,T;H^1_{rad}(B)) \cap L^\infty(B_T)$ such that $\frac{\zeta}{r} \in L^2(0,T;L^2_{rad}(B))$ which satisfy $\zeta = 0$ on $(0,T) \times \{0,1\}$.

- Based on proof by Elliott and Garcke 1996.
- Work in progress: Generalizations. For $n < 4$, $\{|u| = 1\}$ has zero measure.

What happens if $u \to 1$ ($v \to 0$) in finite time?

We will work from now on with $v(r,t) = 1 - u(r,t)$, which satisfies

$$\partial_t v = \frac{1}{r}\partial_r\left(r v^n (2 - v)^n \partial_r\left(-\varepsilon^2 \frac{1}{r}\partial_r(r\partial_r v) + 2(v^3 - 3v^2 + 2v)\right)\right).$$

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Proposition

Let $1 \le n \in \mathbb{N}$ and $v(r,t) > 0$ for all $(r,t) \in (0,1) \times (0,t^*)$ be a smooth solution. If there exists $t^* \in \mathbb{R}$ such that

$$\lim_{t \to t^*} \min_{r \in (0,1)} v(r,t) = \lim_{t \to t^*} v(\bar{r}(t),t) = 0.$$

Then v becomes singular at that point in the following sense:

$$\int_0^{t^*} \left| \partial_{rrr} v(r,t) + \partial_{rr} v(r,t) + \partial_r v(r,t) \right|_{r=\bar{r}(t)} dt = \infty.$$

What happens if $u \to 1$ $(v \to 0)$ in finite time?

We will work from now on with $v(r,t) = 1 - u(r,t)$, which satisfies

$$\partial_t v = -\frac{1}{r}\partial_r\left(r v^n (2-v)^n \partial_r\left(\varepsilon^2 \frac{1}{r}\partial_r(r\,\partial_r v) - 2(v^3 - 3v^2 + 2v)\right)\right).$$

Proposition

Let $1 \leq n < \infty$ and $v(r,t) > 0$ for all $(r,t) \in (0,1) \times (0,t^*)$ be a smooth solution. If there exists $t^* < \infty$ such that

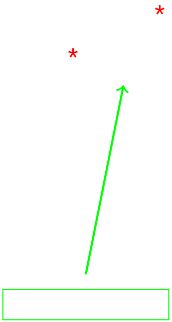
$$\lim_{t \to t^*} \min_{r \in (0,1)} v(r,t) = \lim_{t \to t^*} v(\bar{r}(t),t) = 0.$$

Then v becomes singular at that point in the following sense:

$$\int_0^{t^*} \left| \partial_{rrr} v(r,t) + \partial_{rr} v(r,t) - \partial_r v(r,t) \right|_{r = \bar{r}(t)} dt = \infty.$$

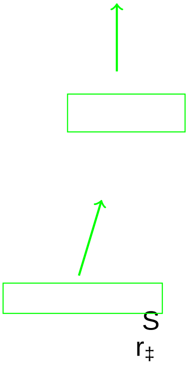
- Following similar thin-film results by Constantin et. al. 2018 and Bertozzi et. al. 1994.

What happens as t ª ? Numerical solution for n 0



Here V 1 U, U is the solution to the constant mobility stationary problem.

What happens as t ª ? Numerical solution for n 4



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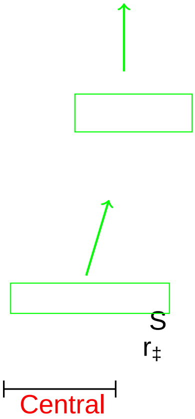

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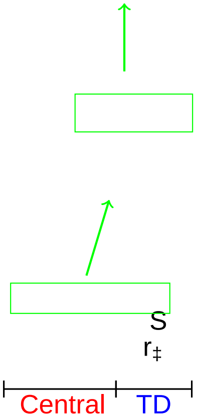

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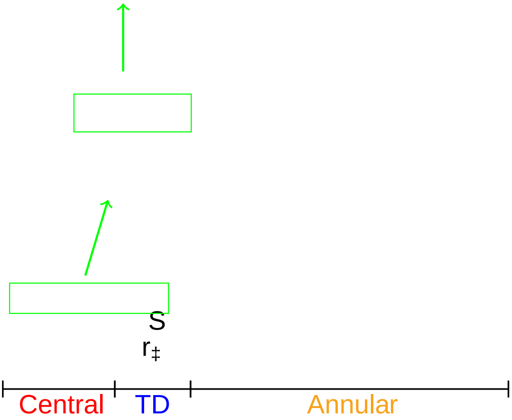

Infinite time touchdown: Case n A 2

Pesce and Münch 2021: We can use matched asymptotics to obtain an asymptotic composite expansion with infinite time touchdown, namely

$$v_{comp}(r;t) \quad v_{central}(r;t) \quad v_{touchdown}(r;t) \quad v_{annular}(r;t)$$
$$A\; t^{\frac{1}{2(n\;1)}}(r \quad r_{\ddagger}) \quad A\,(r \quad r_{\ddagger})^{2};$$

where A ; A are constants fixed by matching.

Pesce and Münch 2021: We can use matched asymptotics to obtain an asymptotic composite expansion with infinite time touchdown, namely

$$v_{comp}(r;t) \sim v_{central}(r;t) + v_{touchdown}(r;t) + v_{annular}(r;t)$$
$$- A\, t^{\frac{1}{2(n-1)}}(r - r_\ddagger) - A(r - r_\ddagger)^2;$$

where A, A are constants fixed by matching.

Matching at leading order gives

$$v_{central} \sim t^{\frac{1}{2(n-1)}}\,\Phi_0(r); \qquad r \in (0; r_\ddagger);$$
$$v_{touchdown} \sim t^{\frac{1}{n-1}}\,\varphi_0(\xi); \quad \xi = \frac{r - r_\ddagger}{t^{\frac{1}{2(n-1)}}}; \qquad \xi \in (-\infty; \infty);$$
$$v_{annular} \sim 1 - U_\ddagger(r); \qquad r \in (r_\ddagger; 1);$$

where Φ_0, φ_0 and $U_\ddagger$ solve ODEs.

PhD Thesis, Pesce 2022: We can find consistent asymptotic expansions only for $1/2 < n \le 2$. Similar to the previous case but now

$$v_{comp}(r,t) \quad v_{central}(r,t) \quad v_{touchdown}(r,t) \quad v_{annular}(r,t)$$
$$A \; t^{\frac{1}{n}}(r \quad r_*)^{\frac{3}{n \; 1}} \quad A \; (r \quad r_*)^2;$$

where A ; A are constants fixed by matching.

In nite time touchdown: Case n B 2

PhD Thesis, Pesce 2022: We can nd consistent asymptotic expansions only for 1~2 @n B 2. Similar to the previous case but now

$$v_{comp}(r;t) \quad v_{central}(r;t) \quad v_{touchdown}(r;t) \quad v_{annular}(r;t)$$

$$A \; t^{\frac{1}{n}}(r \quad r_*)^{\frac{3}{n \; 1}} \quad A \; (r \quad r_*)^2;$$

where A ; A are constants xed by matching.

Matching at leading order gives

$$v_{central} \quad t^{\frac{1}{n}} \; _0(r); \qquad r \in (0; r_*);$$

$$v_{touchdown} \quad t^{\frac{2(n \; 1)}{n(2n \; 1)}} \varphi_0(\;); \qquad \frac{r \quad r_*}{t^{\frac{(n \; 1)}{n(2n \; 1)}}}; \qquad \in (^a \; ; ^a \;);$$

$$v_{annular} \quad 1 \quad U_*(r); \qquad r \in (r_*; 1);$$

where $_0$, φ_0 and U_* solve ODEs.

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Conclusion

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Conclusion

In the setting of radial solutions in 2D unitary ball:

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- Informed by research on thin- lm equations .

Theorem (Rellich-Kondrachov Compactness Theorem)

Assume U is a bounded open subset of $\mathbb{R}^N$, ∂U is C^1 and let $N > mp \geq 1$. If $1 \leq q < \frac{pN}{N-mp}$, then the embedding

$$W^{m,p}(U) \hookrightarrow L^q(U)$$

is compact.

Theorem (Guedes et. al. 2011)

Let $N > mp$ and $\alpha > 0$. If $1 \leq q < \frac{p(N+\alpha)}{N-mp}$, then the embedding

$$W^{m,p}_{rad}(B) \hookrightarrow L^q(B; |x|^{\alpha})$$

is compact.

Proposition (PhD thesis, Pesce 2022)

Let N mp and C0. Then the embedding

$$W^{m;p}_{rad}\hat{}B\bullet\ 0\ L^{q}\hat{}B;\ SxS\bullet$$

is compact for all 1 B q @ª .

Taking 1, m 1 and p 2, we obtain

$$H^{1}_{rad}\hat{}B\bullet\ 0\ L^{q}_{rad}\hat{}B\bullet$$

is compact for all 1 B q @ª . In particular, we take q 2.

Central region:

$$"^2 ‹ @_{rr} \;{}_0ˆr• \quad \frac{1}{r} @_r \;{}_0ˆr•• \quad 4 \;{}_0ˆr• \quad c_1;$$

$$@_r \;{}_0ˆ0• \quad 0;$$

$${}_0ˆr_‡• \quad 0;$$

where c_1 is a constant.

Annular region:

$$\frac{"^2}{r} \frac{d}{dr} ‹ r \frac{dU_‡}{dr}• \quad f^{œ} U_‡• \quad ;$$

$$U_‡^{œ}ˆ1• \quad 0;$$

$$U_‡ˆr_‡• \quad 1; \quad U_‡^{œ}ˆr_‡• \quad 0;$$

where is a constant.

Case $n > 2$

Touchdown region:

$$'^{n}_{0}(\ )@\quad '_0(\ ) = J; \qquad \square\ (-\infty; \infty);$$

$$'_0(\ ) = \begin{cases} A_- \quad + \frac{JA_-^{-n}(-\ \ )^{3-n}}{(n-1)(n-2)(n-3)} + B_- + h.o.t. & \text{if } n < 3; \\ A_- \quad + \frac{J}{2A_-^3}\ln(-\ \ ) + B_- + h.o.t. & \text{if } n = 3; \\ A_- \quad + B_- + h.o.t. & \text{if } n > 3; \end{cases} \text{ as } \quad \to -\infty;$$

$$'_0(\ ) = A_+ \quad ^2 + B_+ \quad + C_+ + h.o.t. \text{ as } \quad \to \infty$$

where $A_\pm$, $B_\pm$, C_+, J are constants.

Central region:

$$-\frac{1}{n}\,{}_0 = -\frac{2^n}{r}\partial_r\left[r\,{}_0^n\partial_r\left(\varepsilon^2\left(\partial_{rr}\,{}_0 + \frac{1}{r}\partial_r\,{}_0\right) - 4\,{}_0\right)\right] \text{ in } (0; r_\square);$$

$${}_0(r) = a_0^- + a_2^- r^2 + h.o.t.; \quad \text{as } r \to 0;$$

$${}_0(r) = a_0^+(r_\square - r)^{\frac{3}{n+1}} + a_1^+(r_\square - r)^{\frac{4n+1+\sqrt{-8n^2+20n+1}}{2(n+1)}} + h.o.t.; \quad \text{as } r \to r_\square;$$

where a_0^-, a_2^-, a_0^+, a_1^+ are constants.

Touchdown region:

$$'^{n}_{0}(\ )@ \qquad '_{0}(\ ) = J; \qquad \square\,(-\infty;\infty);$$

as $\ \to +\infty$, let $n_{\square} \square= \frac{7+3\sqrt{3}}{11}$ we have

$$'_{0}(\ ) = \begin{cases} A_{-}(-\ )^{\frac{3}{(n+1)}} + B_{-}(-\ )^{\frac{2-n}{n+1}} + C_{-}(-\ )^{\frac{4n+1-\sqrt{-8n^2+20n+1}}{2(n+1)}} + h{:}o{:}t{:} & \frac{1}{2} < n < n_{\square}; \\ A_{-}(-\ )^{2-\frac{\sqrt{3}}{3}} + B_{-}x^{1-\frac{\sqrt{3}}{3}} \ln\left(\frac{1}{x}\right) + C_{-}x^{1-\frac{\sqrt{3}}{3}} + h{:}o{:}t{:} & n = n_{\square}; \\ A_{-}(-\ )^{\frac{3}{(n+1)}} + B_{-}(-\ )^{\frac{4n+1-\sqrt{-8n^2+20n+1}}{2(n+1)}} + C_{-}(-\ )^{\frac{2-n}{n+1}} + h{:}o{:}t{:} & n_{\square} < n < 2; \\ ? & n = 2; \end{cases}$$

where

$$A_{-} = \square\frac{-J(n+1)^3}{3(1-2n)(2-n)}\square^{\frac{1}{n+1}}\ :$$

On the other hand, as $\to +\infty$, we have

$$
{}'_0 = \begin{cases}
A_+ \ {}^2 + \frac{-J \ {}^{3-2n}}{A_+^n(2n-1)(2n-2)(2n-3)} + B_+ \ + C_+; & \frac{1}{2} < n < 1; \\
A_+ \ {}^2 + \frac{-J}{A_+} \ (\ln(\ ) + 1) + B_+ \ + C_+; & n = 1; \\
A_+ \ {}^2 + B_+ \ + \frac{-J \ {}^{3-2n}}{A_+^n(2n-1)(2n-2)(2n-3)} + C_+; & 1 < n < \frac{3}{2}; \\
A_+ \ {}^2 + B_+ \ + \frac{J}{2A_+^{3/2}} \ln(\ ) + C_+; & n = \frac{3}{2}; \\
A_+ \ {}^2 + B_+ \ + C_+ + \frac{-J \ {}^{3-2n}}{A_+^n(2n-1)(2n-2)(2n-3)}; & \frac{3}{2} < n \leq 2;
\end{cases}
$$

where $A_\pm$, $B_\pm$, C_+, J are constants.

In the central region, we specifically make the ansatz

$$v(r; t) \,\square\, t \quad (r)$$

with some $\quad < 0$.

This assumption can be tested by plotting $v(r; t)/v(0; t)$ for different times, we expect all curves to collapse near $r = 0$.

Similarly, in the touchdown region,

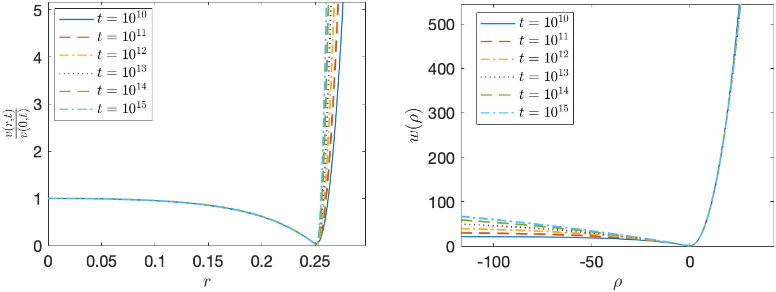
$$v(r;t) \sim t \;\; '(\;\;); \qquad := \frac{r - r_{*}}{t};$$

for some $\;,\; < 0$. We test this ansatz by first scaling

$$w := \frac{v(r;t)}{\min_{r \in [0;1]} v(t)}; \qquad := \left(\frac{\partial_{rr} v(r_{*};t)}{v(r_{*};t)}\right)^{1/2} (r - \bar{r}(t)):$$

Note that $\;\;\sim\;\;$ when t is large.

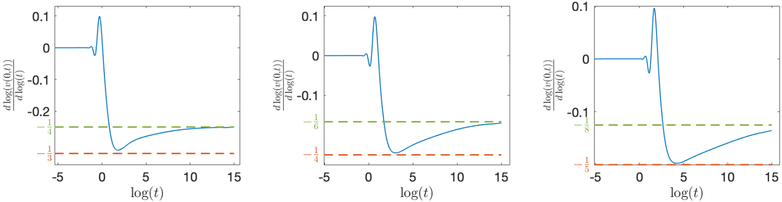
This assumption can be tested by plotting w as a function of $\;$ for different times, we expect all curves to collapse near $r = r_{*}$.



Left: Central region rescaled according to r vs. $v(r;t)/v(0;t)$ for di erent times. Right: Rescaled touchdown region, w vs . For $n = 4$.

To obtain the coe cients we note that, for example in the central region,

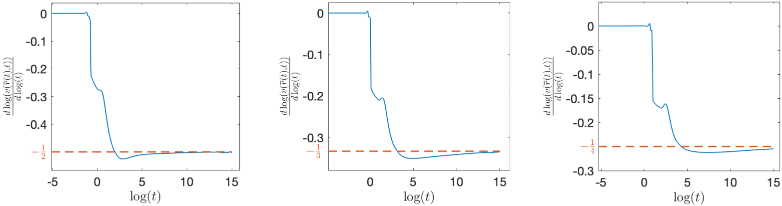
$$\log(v(0;t)) \;\square\; \log(\quad(0)) + \quad \log(t):$$



$\frac{d\log(v(0;t)))}{d\log(t)}$ vs $\log(t)$ for nal time 10^{15} and (left) $n = 3$, (middle) $n = 4$, (right) $n = 5$.

$$\square\; -\frac{1}{2(n-1)}$$

Same for :



$\frac{d\log(v(\bar{r}(t);t)))}{d\log(t)}$ vs $\log(t)$ for nal time 10^{15} and (left) $n = 3$, (middle) $n = 4$, (right) $n = 5$.

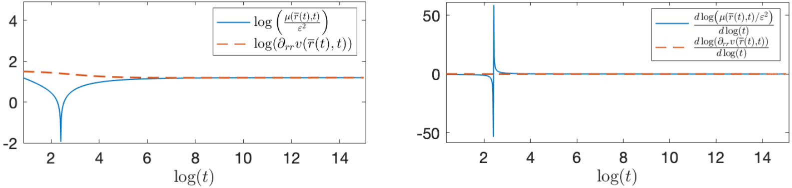
$$□ -\frac{1}{(n-1)}$$

For note

$$@_{rr} v(r_{□}; t) □ t^{-2} @ \quad '( ):$$

Moreover, when t is large

$$(r; t) □ "^2 @_{rr} v(r; t):$$



Left: Log-log for $(\bar{r}; t)/"^2$ and $@_{rr} v(\bar{r}; t)$, Right: Derivative of (left) for $n = 4$ and $" = 0{:}05$.

$$2 \quad □ \quad □ - \frac{1}{(n-1)}$$