# The reduced Ostrovsky equation: integrability and wave breaking

Ted Johnson,
with Karl Helfrich and Roger Grimshaw

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# The reduced Ostrovsky equation

KdV with weak rotation: Ostrovsky equation

$$u_t + \mu u u_x + \lambda u_{xxx} = \gamma v, \qquad v_x = \gamma u.$$

- $\mu$ nonlinearity; $\lambda$ non-hydrostatic; $\gamma$ rotation
- $\lambda = 0$ and $\gamma = 0$ (non-rotating, hydrostatic)
  Inviscid Burgers (Hopf) equation
  All localised or periodic solutions break
- $\gamma = 0$ and $\lambda \neq 0$ (non-rotating, non-hydrostatic): KdV
  No regular initial conditions break
- $\lambda = 0$ and $\gamma \neq 0$ (rotating, hydrostatic)
  Reduced Ostrovsky (Hunter, Vakhnenko) equation.
  Some initial conditions break, others do not

# The reduced Ostrovsky equation

Rescale equation ($\mu = 1$, $\gamma = 1$). Introduce anti-differentiation operator for localised or periodic initial data

$$\partial_x^{-1} u = \int^x u(x', t)\mathrm{d}x',$$

with integration constant chosen so integral over domain or period vanishes (to satisfy zero-mass constraint). Then

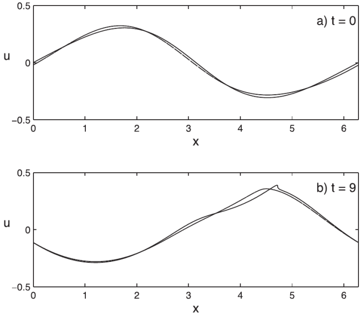
$$u_t + uu_x = \partial_x^{-1} u, \tag{1}$$

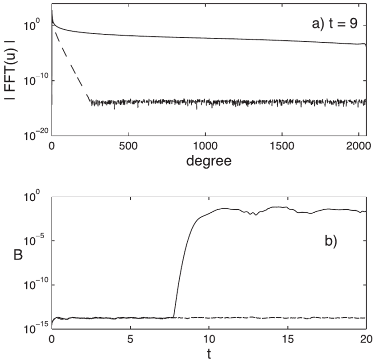
the reduced Ostrovsky equation.

# Previous work

Hunter (1990)
Vakhnenko (1992)
Parkes (1993)
Vakhnenko and Parkes (1998)
Boyd (2004, 2005) (microbreaking)
Stepanyants (2006)
Esler, Rump & Johnson (2009)
Liu *et al* (2010)
Kraenkel *et al* (2011)

# Microbreaking



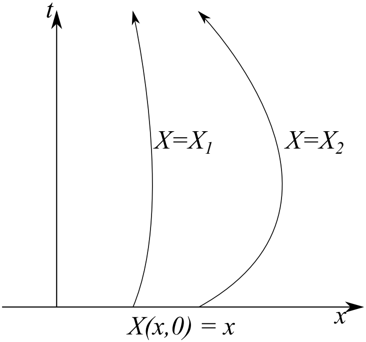



B: average of magnitude of highest 128 of 2048 Fourier coefficients.

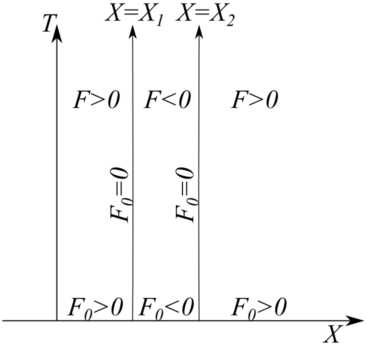
## Characteristic co-ordinates

- The RedO
$$u_t + uu_x = \partial_x^{-1} u, \qquad (2)$$
is a quasi-linear first-order pde with one set of characteristics.
- On characteristics
$$\frac{\mathrm{d}x}{\mathrm{d}t} = u, \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \partial_x^{-1} u.$$
- Let the characteristics be the lines $\mathcal{X}(x,t) = \mathrm{constant}$. Lagrangian co-ordinate (Zeitlin *et al.* 2003, 1D rSWE).
- In terms of $(\mathcal{X}, T)$ with $t = T$ and $u(x,t) = U(\mathcal{X}, T)$
$$x_T = U, \qquad U_T = \partial_x^{-1} U,$$
with $\mathcal{X} = x$ at $T = 0$.

# Characteristic co-ordinates




Laboratory frame




Characteristic (Lagrangian) frame

## Characteristic co-ordinates

Our system is thus

$$x_T = U, \qquad U_T = \partial_x^{-1} U,$$

with $\mathcal{X} = x$ at $T = 0$.

Differentiating wrt $\mathcal{X}$ gives the pair

$$x_{\mathcal{X}T} = U_{\mathcal{X}}, \qquad U_{\mathcal{X}T} = \partial_{\mathcal{X}} \partial_x^{-1} U = x_{\mathcal{X}} \partial_x \partial_x^{-1} U = x_{\mathcal{X}} U,$$

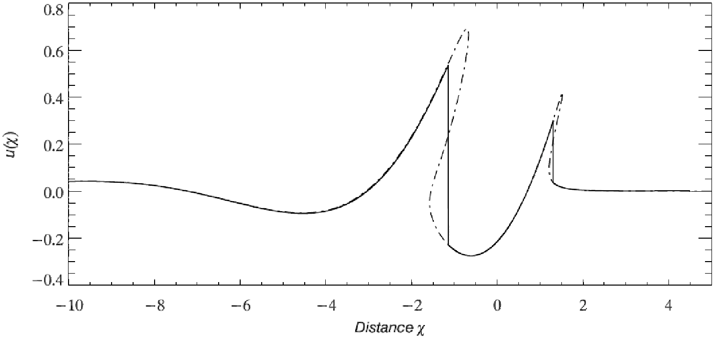
i.e.

$$\phi_T = W, \qquad W_T = \phi U,$$

where $W = U_{\mathcal{X}}$ and $\phi = x_{\mathcal{X}}$ is the Jacobian of the transformation to characteristic co-ordinates.

# The Jacobian, $\phi$

- $\phi$ is initially unity
- Provided $\phi$ remains bounded and positive the transformation is 1:1 and the wave does not break.
- If $\phi$ passes through zero then the waves overturns (breaks). (Nothing untoward numerically).



# Kraenkel *et al* (2011)

Differentiating (1) w.r.t. $x$ twice and rearranging gives

$$F_t + (uF)_x = 0\,.$$

where

$$F^3 = 1 - 3u_{xx}\,.$$

i.e. $F$ is a conserved density.

## The density $F = (1 - 3u_{xx})^{1/3}$ in characteristic co-ordinates

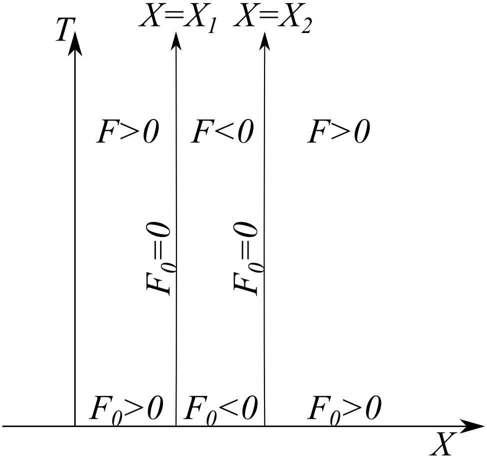
$$(F\phi)_T = 0\,, \quad \text{so that} \quad F\phi = F_0(\mathcal{X}),$$

$$\text{where} \qquad F_0(\mathcal{X}) = F(\mathcal{X}, 0) = F(x, 0),$$

determined by the initial conditions.

- Until breaking $\phi > 0$. Thus $F(\mathcal{X}, T) = F_0(\mathcal{X})/\phi(\mathcal{X}, T)$
- On each characteristic
  - If $F_0(\mathcal{X}) > 0$, then $F(\mathcal{X}, T) > 0, \quad \forall T \geq 0$.
  - If $F_0(\mathcal{X}) < 0$, then $F(\mathcal{X}, T) < 0, \quad \forall T \geq 0$.
  - If $F_0(\mathcal{X}) = 0$, then $F(\mathcal{X}, T) = 0, \quad \forall T \geq 0$.
- Until breaking, the $\mathcal{X}$-domain is permanently divided by the initial conditions into $\mathcal{X}$-intervals where $F > 0$ and the remaining $\mathcal{X}$-intervals where $F < 0$.

# The density $F$ in characteristic co-ordinates




Characteristic (Lagrangian) frame

## Reduction of order, $F = (1 - 3u_{xx})^{1/3}$

Now $$u_{xx} = \frac{1}{\phi}\{\frac{U_{\mathcal{X}}}{\phi}\}_{\mathcal{X}} = \frac{1}{\phi}\{\frac{\phi_T}{\phi}\}_{\mathcal{X}} = \frac{\{\log\phi\}_{\mathcal{X}T}}{\phi},$$

i.e. $$F^3 = 1 - (3/\phi)\{\log\phi\}_{\mathcal{X}T}.$$

Combining this with $F\phi = F_0(\mathcal{X})$ gives

$$(\log\phi)_{\mathcal{X}T} = \frac{\phi}{3}(1 - \frac{F_0^3}{\phi^3})\,, \tag{3}$$

or $$(\log F)_{\mathcal{X}T} = \frac{F_0}{3F}(F^3 - 1)\,, \tag{4}$$

equations for $\phi$ and $F$ alone.

## Integrability: $F_0(\mathcal{X}) > 0 \; \forall \mathcal{X}$, $\qquad F = (1 - 3u_{xx})^{1/3}$

following Kraenkel *et al.*(2011)

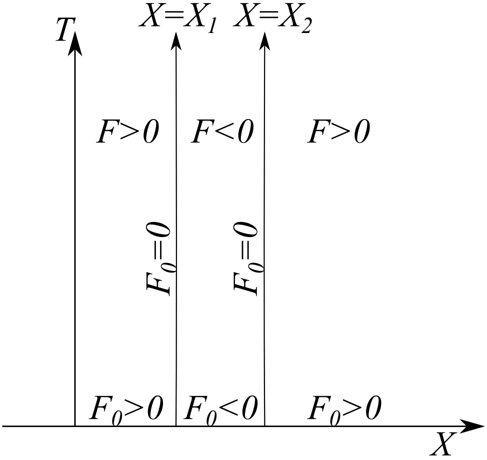
- For smooth bounded initial conditions $u_{xx} = 0$ somewhere.
- Thus $F_0(\mathcal{X}) = 1$ for some $\mathcal{X}$.
- Thus suppose $F_0(\mathcal{X}) > 0 \; \forall \mathcal{X}$ at $T = 0$.
- Introduce $\zeta$ through the 1:1 mapping defined by
  $$\mathrm{d}\zeta = (1/3) F_0(\mathcal{X}) \, \mathrm{d}\mathcal{X}.$$
- Then equations (3),(4) reduce to the *integrable* Tzitzeica (1910) equation
  $$(\log h)_{\zeta T} = h - h^{-2},$$
  where $h = \phi / F_0 = 1/F$. (Kraenkel *et al.* : Dodd-Bullough, 1977, equation)
- $h > 0 \; \forall T$ so $\phi > 0 \; \forall T$

# Integrability

- Hence the RedO (1) is integrable for initial data such that $F_0 > 0$
- i.e. if $u_{xx} < 1/3$ everywhere at any instant (including $t = 0$), then the interface evolves for all time without breaking (and such that $u_{xx} < 1/3$ everywhere)
- This remains true even if $F_0(\mathcal{X})$ vanishes at isolated values of $\mathcal{X}$ (since the transformation to $\zeta$ remains 1:1).
- Now suppose there exists an interval $x_1 \leq x \leq x_2$ in which $u_{0xx} \geq 1/3$, with equality only at the end points. Then $F_0(x) \leq 0$ so

$$F(\mathcal{X}, T) < 0, \quad \forall \mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2, \quad \forall T \geq 0.$$

## $F$ negative in an interval




Characteristic (Lagrangian) frame

# The interval $\mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2$, $F_0(\mathcal{X}) < 0$

- Integrating equation (3) for $\phi$ in time (i.e. wrt $T$) gives

$$\beta(\mathcal{X}, T) = (\log \phi)_{\mathcal{X}} = \int_0^T \frac{\phi}{3}(1 - \frac{F_0^3}{\phi^3})\, dT \,. \tag{5}$$

- The integrand is positive for all $\phi > 0$, with a minimum value of $-2^{-2/3} F_0(\mathcal{X})$ achieved where $\phi = -2^{1/3} F_0(\mathcal{X})$, independently of $T$.
- Thus $\beta > 0$ in $\mathcal{X}_1 \leq \mathcal{X} \leq \mathcal{X}_2$. So $\phi_{\mathcal{X}} > 0$ there. So $\phi$ cannot achieve a minimum value in this interval.
- Thus breaking (if it occurs) occurs first at a point corresponding to $u_{xx} < 1/3$ in initial data (the integrable region).

## Breaking

- Now, for each $\mathcal{X}$ in the interval $\mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2$,

$$\beta = (\log\phi)_{\mathcal{X}} > -2^{-2/3} F_0(\mathcal{X})\, T\,,$$

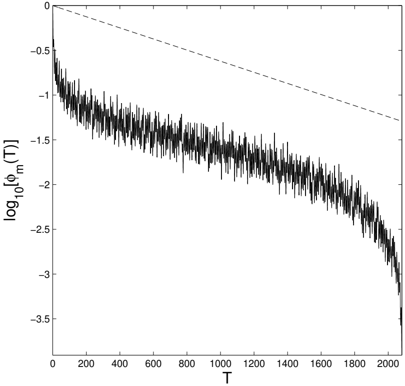
- Integrating over the interval $\mathcal{X}_1 < \mathcal{X} < \mathcal{X}_2$ yields

$$\phi(\mathcal{X}_1, T) < \phi(\mathcal{X}_2, T) \exp\left(-\alpha T\right),$$

$$\alpha = 2^{-2/3} \int_{\mathcal{X}_1}^{\mathcal{X}_2} \left(-F_0(\mathcal{X})\right) d\mathcal{X} = 2^{-2/3} \int_{x_1}^{x_2} \left\{3u_{0xx}(x) - 1\right\}^{1/3} dx\,.$$

- Thus the Jacobian $\phi(\mathcal{X}_1, T)$ at the left-hand end of the interval on which $F_0$ is negative becomes exponentially small compared to its value $\phi(\mathcal{X}_2, T)$ at the right-hand end.

# Jacobian minimum

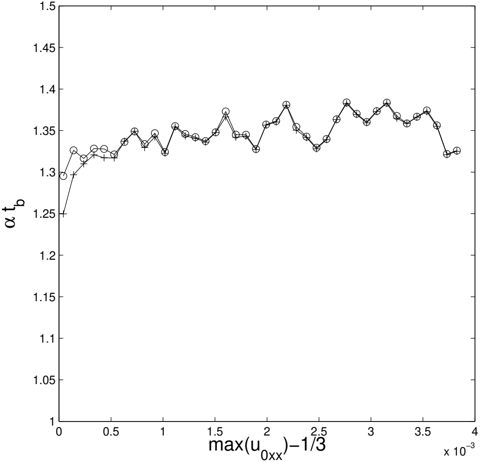



The logarithm of the minimum, $\phi_m(T)$, over $\mathcal{X}$ of the Jacobian $\phi(\mathcal{X}, T)$ as a function of $T$ for the initial profile

$$u_0(x) = u_1 \sin(x) + u_2 \sin(2x + \theta),$$

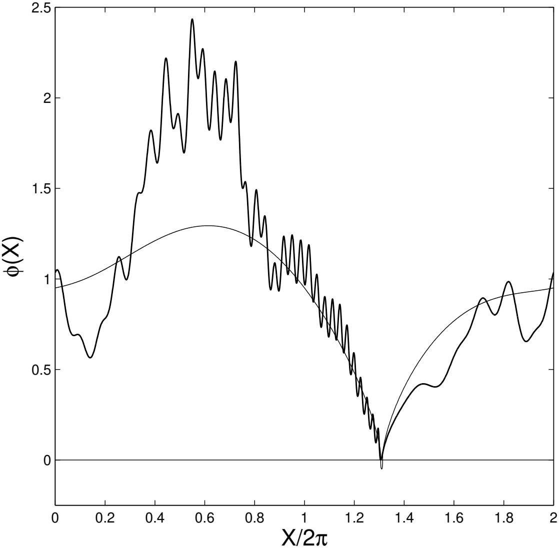
(where $\theta$ is an arbitrary phase shift). Here $u_1 = 0.3$, $u_2 = 0.03$ and $\theta_0 = 3.5453$ so $\max(u_{0xx}) - 1/3 = 4 \times 10^{-5}$, computed with $N = 4096$ nodes. The wave breaks when $\phi_m$ first vanishes, at $T = t_b = 2081.7$.

# Breaking-time scaling



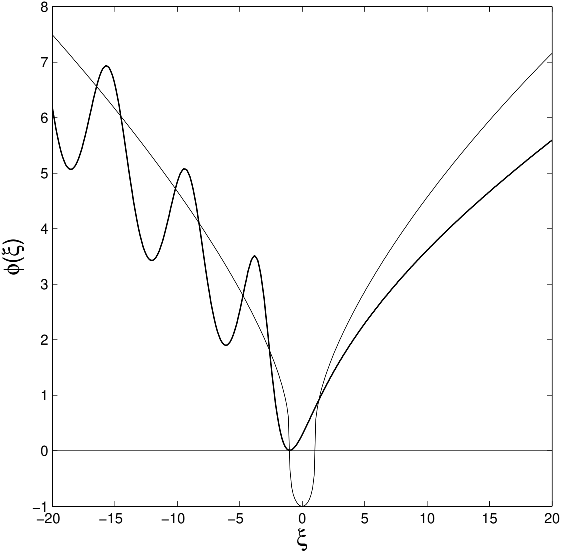

The scaled time to breaking, $\alpha t_b$, for this initial profile for varying $\theta_0$ as a function of the excess of $u_{0xx}$ over $1/3$. The number of nodes in the computations are: '$+$' $N = 2048$ and 'o' $N = 4096$.

# Jacobian at breaking




The Jacobian $\phi(\mathcal{X}, t_b)$ at the instant of breaking. The thinner curve shows $F_0(\mathcal{X})$ which is negative in a region surrounding $1.31\pi$.

# Jacobian at breaking - detail




The scaled Jacobian $\phi(\hat{\xi})$ as a function of the scaled co-ordinate $\xi$. The scaling is such that the region of negative $F_0(\mathcal{X})$ has unit depth and width 2.

# Jacobian at breaking - asymptotic form

- Consider a weakly supercritical initial condition where $u_{0xx}$ is smooth with maximum at $\mathcal{X}_0$ slightly exceeding $1/3$.
- Near $\mathcal{X}_0$,
$$u_{0xx} = a - b(\mathcal{X} - \mathcal{X}_0)^2 + \cdots ,$$
where $a = \max(u_{0xx}) = u_{0xx}(\mathcal{X}_0)$ and $b = -(1/2)u_{0xxxx}(\mathcal{X}_0) > 0$.

- Then
$$[F_0(\mathcal{X})]^3 = (3a-1)[-1 + \xi^2 + \cdots],$$
where $\xi = (\mathcal{X} - \mathcal{X}_0)[3b/(3a-1)]^{1/2}$ and $\xi = \pm 1$ corresponds to $\mathcal{X} = \mathcal{X}_2, \mathcal{X}_1$ in the general problem.

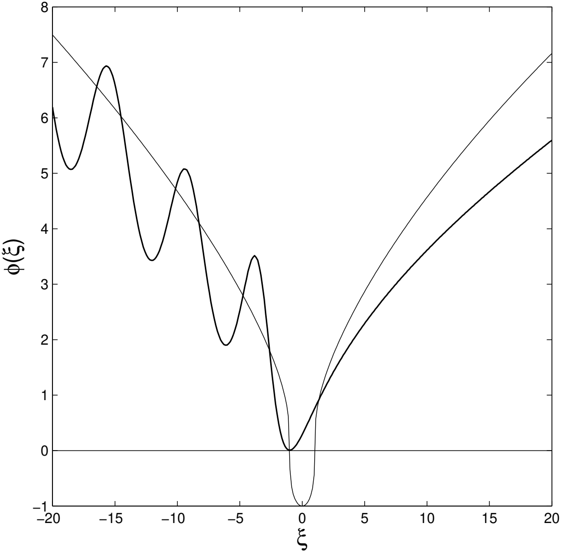
- Write
$$\phi = (3a-1)^{1/3}\hat{\phi},$$
giving the *parameter-free* generic equation near breaking,
$$(\log \hat{\phi})_{\xi T} = (\hat{\phi}/3)[1 + (1-\xi^2)/\hat{\phi}^3],$$
where $T = \epsilon\tau$ for $\epsilon = (3a-1)^{5/6}/\sqrt{3b}$.

- *The time to breaking scales as* $[\max(u_{0xx}) - 1/3]^{5/6}$.

# Jacobian minimum at large time

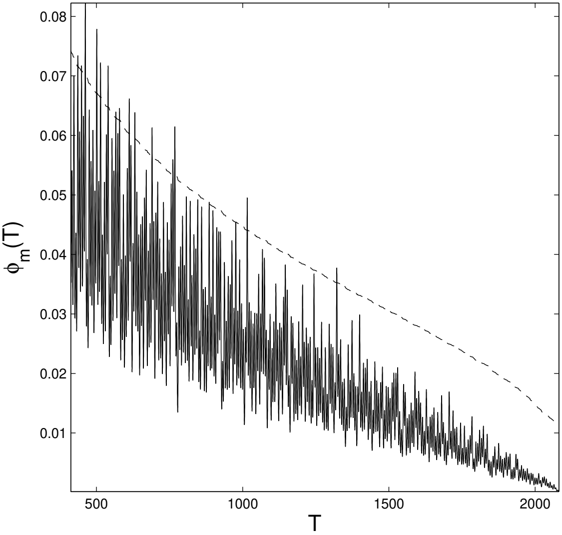
- Dropping the first term in the governing equation (less than 1/8th the second) gives
$$(\log \phi)_{\mathcal{X}T} = -(1/3)F_0^3/\phi^2.$$
- This has solution
$$\phi = A + B(\mathcal{X})T,$$
for $A$ constant and $B(\mathcal{X})$ a function of $\mathcal{X}$ alone, provided $AB'(\mathcal{X}) = -(1/3)F_0^3$.
- Near breaking
$$\phi = A(1 - t/t_b) + (t/3A)\int_{\mathcal{X}}^{\mathcal{X}_1} F_0^3(\mathcal{X}')\mathrm{d}\mathcal{X}'.$$
- Since $F_0 > 0$ in $\mathcal{X} < \mathcal{X}_1$ and $F_0 < 0$ in $\mathcal{X} > \mathcal{X}_1$ this gives $\phi$ increasing monotonically with distance from a local minimum at $\mathcal{X} = \mathcal{X}_1$ of
$$\phi_m = A(1 - t/t_b).$$
- The Jacobian does indeed appear to decrease linearly with $t$ at large $t$ until vanishing at $t_b$.

# Jacobian at breaking - detail




The scaled Jacobian $\phi(\hat{\xi})$ as a function of the scaled co-ordinate $\xi$. The scaling is such that the region of negative $F_0(\mathcal{X})$ has unit depth and width 2.

# Jacobian minimum at large time




The minimum of the Jacobian, $\phi_m(T)$, as a function of time for $T > 400$. The dashed line shows the corresponding value of $F_0$ at the same $\mathcal{X}$ and $T$, i.e. $F_m(T) = F_0(\mathcal{X}_m(T))$. Note that at large $T$, $\phi_m$ is less than $\frac{1}{2}F_m$.

# Ostrovsky number

- In the unscaled equation an Ostrovsky number can be defined as

$$O_s = 3\mu\kappa/\gamma^2\,, \quad \text{where} \quad \kappa = \max[u_{0xx}(x)]\,.$$

- Initial conditions with $O_s > 1$ break and those with $O_s \leq 1$ do not.
- Increasing nonlinearity ($\mu$) or curvature ($\kappa$) increases $O_s$.
- Increasing rotation ($\gamma$) decreases $O_s$.

# Modified reduced Ostrovsky equation

- A rotating, hydrostatic, two-layer, Boussinesq fluid where the layers have equal depths, is governed by the mRO

$$u_t + (1/2)u^2 u_x = \partial_x^{-1} u.$$

- Similar considerations show that
  - If $|u_{0x}| < 1$ everywhere, the wave never breaks.
  - If $|u_{0x}| > 1$ somewhere, the wave breaks in finite time.

# Orbital stability of periodic solutions

Travelling $2L$-periodic solutions of the reduced Ostrovsky equation have the normalized form

$$u(x,t) = \frac{L^2}{\pi^2} U(z), \quad z = \frac{\pi}{L} x - \frac{L}{\pi} \gamma t,$$

where $U(z)$ is a $2\pi$-periodic solution of the second-order differential equation

$$\frac{d}{dz}\left[(\gamma - U)\frac{dU}{dz}\right] + U(z) = 0,$$

and the parameter $\gamma$ is proportional to the wave speed.

- $U$ has zero mean.
- $U$ can be taken as even in $z$.
- $U$ exists for every $\gamma \in \left(1, \frac{\pi^2}{9}\right)$.
- As $\gamma \to \frac{\pi^2}{9}$ the limiting wave has a (non-smooth) parabolic profile ($F \equiv 0$).

## Lyapunov functional: first try

- Conserved momentum $Q(u) = \|u\|_{L^2}^2$.
- Conserved energy

$$E(u) = \|\partial_x^{-1} u\|_{L^2}^2 + \frac{1}{3}\int u^3 dx,$$

- Introduce the functional

$$S_\gamma(u) := E(u) - \gamma Q(u).$$

- As usual, the Euler–Lagrange equations for $S_\gamma$ gives the redO.

# First try, second variation

- Take $v$ square integrable $2\pi N$-periodic function with zero mean.
- Expand $S_\gamma(U+v) - S_\gamma(U)$ to quadratic order in $v$.
- Obtain second variation

$$\delta^2 S_\gamma = \int \left[ (\partial_z^{-1} v)^2 - (\gamma - U) v^2 \right] dz.$$

- Not sign definite.
- Write this as the quadratic form

$$\delta^2 S_\gamma = \langle L_\gamma v, v \rangle_{L^2},$$

where $L_\gamma$ is the self-adjoint operator

$$L_\gamma := -\partial_z^{-2} - \gamma + U.$$

# Lyapunov functional: second try

There are other conserved quantities of the redO.

- Higher order energy

$$H(u) = \int \frac{(u_{xxx})^2}{(1-3u_{xx})^{7/3}}\,dx,$$

- Casimir-type functional $C(u) = \int (1-3u_{xx})^{1/3} dx$.
- Define a second energy functional $R_\Gamma(u) := C(u) - \Gamma H(u)$,
- Choose parameter $\Gamma$ so the same periodic wave $U$ that is critical point of $S_\gamma$ is a critical point of $R_\Gamma(u)$, then

$$\Gamma := -(\gamma^3 - 6I)^{-2/3},$$

$$I = \frac{1}{2}\left(\gamma - \frac{1}{2}U^2\right)^2 \left(\frac{dU}{dz}\right)^2 + \frac{\gamma}{2}U^2 - \frac{1}{8}U^4 = \text{const.}$$

# Second try, second variation

- $$\delta^2 R_\Gamma := \int \left[ \frac{v^2}{(\gamma^3 - 6I)^{2/3}} - \frac{v_{zz}^2}{(1 - 3U'')^{5/3}} \right] dz.$$
- Not sign definite.
- Write this as the quadratic form

  $$\delta^2 R_\Gamma = \langle M_\gamma v, v \rangle_{L^2},$$

  where $M_\gamma$ is the self-adjoint operator

  $$M_\gamma := -\partial_z^2 (1 - 3U'')^{-5/3} \partial_z^2 + (\gamma^3 - 6I)^{-2/3}.$$

## A linear combination

- Introduce
$$\Lambda_{c,\gamma}(u) := S_\gamma(u) - cR_\Gamma(u),$$
where $c \in \mathbb{R}$ is a parameter to be defined within an appropriate interval.
- We wish to characterize the spectrum of the linear operator $K_{c,\gamma} := L_\gamma - cM_\gamma$.
- $K_{c,\gamma}$ is self-adjoint with $2\pi$-periodic coefficients by construction.
- By Bloch's theorem it is sufficient to seek eigenfunctions of the form
$$e^{i\kappa z} w(z,\kappa)$$
with eigenvalues $\lambda(\kappa)$ where $\kappa$ lies in the Brillouin zone $\mathbb{T} = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $w(z,\kappa)$ is $2\pi N$-periodic.
- Thus introduce the operator
$$P_{c,\gamma}(\kappa) := e^{-i\kappa z} K_{c,\gamma} e^{i\kappa z},$$
and look for its $2\pi N$-periodic eigenfunctions $w(z,\kappa)$ and eigenvalues $\lambda(\kappa)$.

## Numerical treatment of the operator $P_{c,\gamma}(\kappa)$

- Write

$$P_{c,\gamma}(\kappa) = A_\gamma(\kappa) - cB_\gamma(\kappa),$$

where

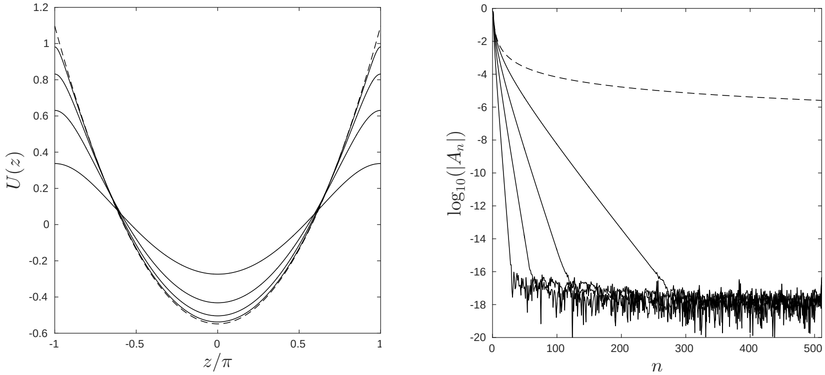
$$\begin{aligned} A_\gamma(\kappa) &= -(\partial_z + i\kappa)^{-2} - (\gamma - U), \\ B_\gamma(\kappa) &= (\gamma^3 - 6I)^{-2/3} - (\gamma^3 - 6I)^{-5/3}(\partial_z + i\kappa)^2(\gamma - U)^5(\partial_z + i\kappa)^2. \end{aligned}$$

- Discretise the linear operators in Fourier space and evaluate products pseudospectrally, so

$$\widehat{A_\gamma}(\kappa) = \mathrm{diag}(\mathbf{k_1}^2) - \mathcal{F}(\mathrm{diag}(\gamma - \mathbf{U})\mathcal{F}^{-1}(\mathbf{I})),$$
$$\widehat{B_\gamma}(\kappa) = (\gamma^3 - 6I)^{-2/3}\mathbf{I} - (\gamma^3 - 6I)^{-5/3}\mathrm{diag}(\mathbf{k}^2)\mathcal{F}(\mathrm{diag}(\gamma - \mathbf{U})^5\mathcal{F}^{-1}(\mathrm{diag}(\mathbf{k}^2)),$$

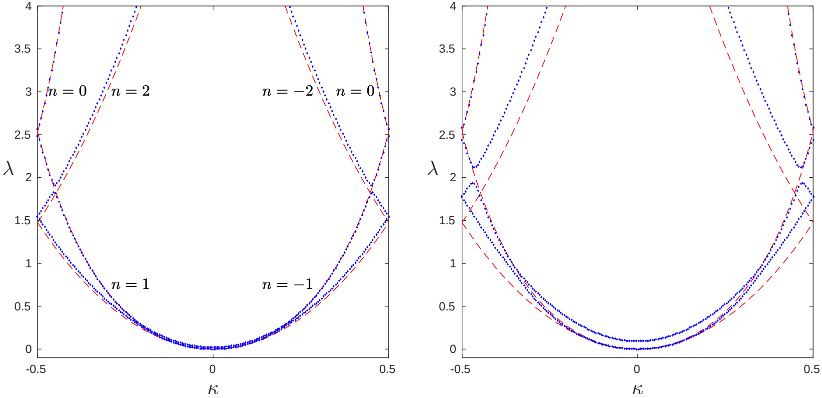
where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the discrete Fourier transform and its inverse, $\mathbf{k}$ is the wavenumber vector with components $\kappa \pm n$ and $\mathbf{k_1}$ its component-wise inverse.
- Eigenvalues obtained using the Matlab subroutines `eig` and `eigs`.

# The base periodic solutions



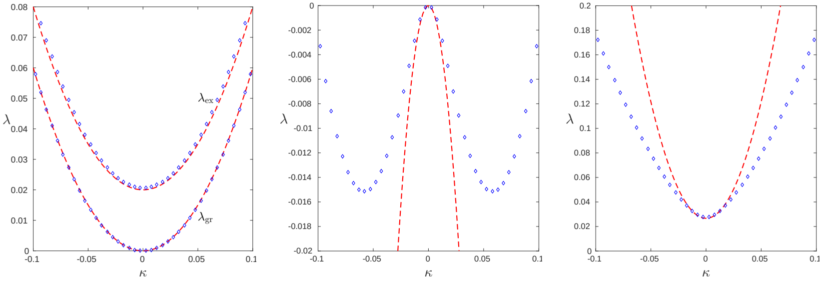

- (a) $2\pi$-periodic solutions of the redO for $a = A_1 = -0.3, -0.5, -0.6, -0.65$.
- (b) Log of the absolute value of the Fourier cosine coefficients, $A_n$.
- Dashed: limiting piecewise parabolic wave ($a = -\frac{2}{3}$) with coefficients $A_n = 2(-1)^n/3n^2$.
- Spectral Newton-Kantorovich iteration on $A_n$, $\gamma$.

# The lowest eigenvalues of the operator $P_{c,\gamma}(\kappa)$ when $c = 0.5$




- Left: $a = -0.1$ and Right : $a = -0.2$.
- Red dashed: the lowest eigenvalues of the unperturbed operator for $a = 0$.
- Blue diamonds: computed eigenvalues.
- All repeated eigenvalues for $a = 0$ are split when $a \neq 0$.
- Thus for $c = 0.5$, $\Lambda_{c,\gamma}(u)$ provides a Lyapunov functional for $a = -0.1$ and $a = -0.2$.

# Small-$\kappa$, small-$a$ asymptotics (dashed red) and numerics




- Left: Detail of previous figure ($c = 0.5$, $a = -0.1$) in neighbourhood of origin. The two spectral bands split at finite $a$.
- Centre: The ground spectral band for $a = -0.1$ but for $c = 0.7$.
- Thus for $c = 0.7$, $\Lambda_{c,\gamma}(u)$ does not provide a Lyapunov functional for $a = -0.1$.
- Right: The first excited spectral band for $a = -0.1$, $c = 0.7$.
- Ground state transition from concave upwards (left) to concave downwards (centre) with increasing $|c|$ is generic.
- At fixed $a$ the graph of the spectral band $\lambda_{\rm gr}(\kappa)$ is concave upwards at $\kappa = 0$ for $c \in (c_-, c_+)$ and concave downwards outside this interval.

# Determining the positivity of $P_{c,\gamma}(\kappa)$

- At fixed $a$ the graph of the spectral band $\lambda_{\mathrm{gr}}(\kappa)$ is concave upwards at $\kappa = 0$ for $c \in (c_-, c_+)$ and concave downwards outside this interval.
- This is first occurrence of a negative eigenvalue of $P_{c,\gamma}(\kappa)$.
- Thus boundaries $c_\pm$ are determined by changes in sign of $\lambda''_{\mathrm{gr}}(0)$.
- Since $\lambda'_{\mathrm{gr}}(0) = 0$, the sign of $\lambda''_{\mathrm{gr}}(0)$ is the sign of $\lambda_{\mathrm{gr}}(\delta_\kappa)$ for $0 < \delta_\kappa \ll 1$.
- $c_\pm$ are thus determined as the values of $c$ for which $P_{c,\gamma}(\delta_\kappa)$ has a zero eigenvalue, i.e. $det[P_{c,\gamma}(\delta_\kappa)] = 0$, i.e.

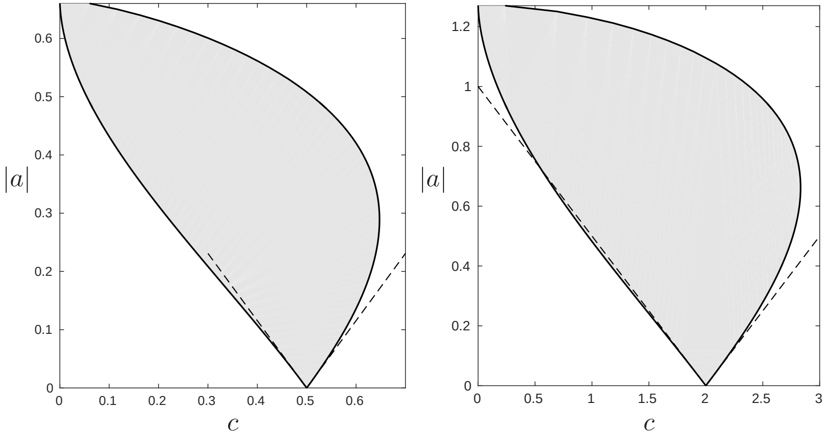
  $$det[A_\gamma(\delta_\kappa) - cB_\gamma(\delta_\kappa)] = 0,$$

  i.e eigenvalues of the generalised linear eigenvalue problem

  $$A_\gamma(\delta_\kappa) = cB_\gamma(\delta_\kappa).$$

- Computations performed for $\delta_\kappa = 10^{-2}, 10^{-3}, 10^{-4}$. Results graphically indistinguishable.

## Region of $(c, |a|)$ plane where $P_{c,\gamma}(\kappa)$ positive $\forall\kappa$




- Left: the reduced Ostrovsky equation
- Right: the modified reduced Ostrovsky equation
- The dashed lines show small $|a|$ expansions for the boundaries

## Conclusions

- Reduced Ostrovsky breaks if $3u_{xxx} > 1$, integrable otherwise.
- For small excesses of $3u_{xxx}$ over 1, breaking time varies as $[\max(u_{0xx}) - 1/3]^{5/6}$.
- Periodic solutions to the reduced Ostrovsky and modified reduced Ostrovsky equations are orbitally stable.

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