Decay estimates for large velocities in the Boltzmann equation without cut-off

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joint work with C. Mouhot (Cambridge) and L. Silvestre (Chicago)

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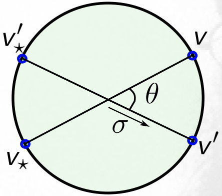
The Boltzmann equation

$$\underbrace{\partial_t f + v \cdot \nabla_x f}_{\textbf{free transport operator}} \quad = \quad \underbrace{Q(f,f)}_{\textbf{collision kernel}} \quad = \int (f'_* f' - f f_*) B(r, \cos\theta)\, \mathrm{d}\, v_* \mathrm{d}\,\sigma$$

where $f'_* = f(v'_*)$, $\quad f' = f(v')$, $\quad f_* = f(v_*)$

Pre- and post-collisional velocities $(v, v_*/v', v'_*)$



$$\begin{cases} v + v_* = v' + v'_* \\ |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2 \end{cases}$$

$$\begin{cases} r = |v - v_*| \\ \cos\theta := \frac{v - v_*}{|v - v_*|} \cdot \frac{v' - v'_*}{|v' - v'_*|} \end{cases}$$

Deviation angle θ

Cross section B: the way "particles" interact

$$Q(f,f) = \int (f'_* f' - f f_*) B(r, \cos\theta) \,\mathrm{d}\, v_* \,\mathrm{d}\,\sigma$$

$$\boxed{B(r,\cos\theta) = r^{\gamma} b(\cos\theta) \quad \text{with} \quad -d < \gamma \le 2}$$

Cut-off $\displaystyle\int_{S^{d-1}} b(\cos\theta)\,\mathrm{d}\,\sigma < \boxed{+\infty}$

Non cut-off $b(\cos\theta) \simeq \boxed{\dfrac{1}{|\theta|^{d-1+2s}}}$ with $s \in (0,1)$

Moment estimates (cut-off)

Moment of order k: $\int_{\mathbb{R}^d} f(t,x,v)|v|^k \mathrm{d}v$

$$\left\{\begin{array}{l}\text{classical \& important theme in kinetic theory}\\ \text{first step in the theory of the Boltzmann equation}\\ \text{mostly for } \gamma > 0 \text{ and for the homogeneous case}\end{array}\right.$$

Moment estimates (cut-off)

Hard potentials ($\gamma > 0$) and homogeneous

- Povzner'62, Elmroth'83
  generation of moments by Povzner's identities
- Desvillettes '93
  if k-moment with $k > 2$ initially, then all moments are generated
- Wennberg '97
  not even necessary to have an initial > 2 moment
- Bobylev '97
  L^1 estimates with exponential weight

Moderately soft potentials ($-2 < \gamma < 0$) and homogeneous

- Desvillettes '93
  propagation of moments

Pointwise decay cannot be generated (cut-off)

Recall $B(r, \cos\theta) = |v_* - v|^\gamma b(\cos\theta)$
$\cos\theta = k \cdot \sigma$ with $k = (v_* - v)/|v_* - v|$

Gain and loss terms

$$
\begin{aligned}
Q(f,f) &= \int (f'_* f' - f f_*) B(r, \cos\theta) \,\mathrm{d}\, v_* \,\mathrm{d}\,\sigma \\
&= \underbrace{\int f'_* f' B(r, \cos\theta) \,\mathrm{d}\, v_* \,\mathrm{d}\,\sigma}_{Q_+(f,f)} - \underbrace{\int f f_* B(r, \cos\theta) \,\mathrm{d}\, v_* \,\mathrm{d}\,\sigma}_{Q_-(f,f)}
\end{aligned}
$$

Pointwise decay cannot be generated (cut-off)

Recall $Q(f,f) = Q_+(f,f) - Q_-(f,f)$

Loss term

$$\begin{aligned} Q_-(f,f) =& f(v)\int f(v_*)|v-v_*|^\gamma \,\mathrm{d}\, v_* \underbrace{\left(\int b(\cos\theta)\mathrm{d}\sigma\right)}_{c_b} \\ =& c_b f (f *_v |\cdot|^\gamma) \\ \leq& C_0 f(1+|v|^\gamma) \end{aligned}$$

for some constant C_0 only depending on $\int f_0(1+|v|^2)\mathrm{d}v$.

A lower bound for f

$$\partial_t f = Q(f,f) = Q_+(f,f) - Q_-(f,f) \geq -C_0(1+|v|^\gamma)f.$$

$\boxed{f \geq e^{-tC_0(1+|v|^\gamma)} f_0}$ $\qquad f_0 =$ initial condition

Generation of pointwise decay (non cut-off)

Non-homogeneous

- Generation of polynomial decay
- Arbitrary polynomial decay for $\gamma > 0$
- Restricted polynomial decay for $\gamma < 0$ (moderately soft)

Conditional to hydrodynamical bounds

- $0 \leq m_0 \leq \int f(t,x,v)\mathrm{d}v \leq M_0$
- $\int f(t,x,v)|v|^2\mathrm{d}v \leq E_0$
- $\int f \ln f(t,x,v)\mathrm{d}v \leq H_0$

Theorem (CI, Mouhot, Silvestre)

Let $\gamma + 2s \in [0,2]$ *and* f *be a solution of the Boltzmann equation such that* • *mass, energy and entropy are controlled* •. *Then*

1. *(Generation)* $\boxed{\text{If } \gamma > 0}$ *then* $\boxed{\forall q > 0}$, $\exists N, \beta > 0$

$$f(t,x,v) \leq N(1 + t^{-\beta}) \min(1, |v|^{-q})$$

2. *(Generation)* $\boxed{\text{If } \gamma \leq 0}$, *then*

$$f(t,x,v) \leq N(1 + t^{-\frac{d}{2s}}) \min\left(1, |v|^{-\boxed{1 + \frac{d(\gamma+2s)}{2s}}}\right)$$

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$$f(t,x,v) \leq N(1 + t^{-\frac{d}{2s}}) \min\left(1, |v|^{-\boxed{1 + \frac{d(\gamma+2s)}{2s}}}\right)$$
3. *(Propagation)* $\boxed{\text{If } \gamma \leq 0}$, *then* $\forall q \geq {}^{\exists} q_0$, $f_{in} \leq C \min(1, |v|^{-q})$
$$f(t,x,v) \leq N \min(1, |v|^{-q})$$

Conditional C^∞ smoothing effect (non-cutoff)

Reduce well-posedness of Boltzmann non-cutoff to hydro controls

This important equation is well-posed as long as mass and energy do not explode.

Conditional regularity for Boltzmann

- L^∞ bound (Silvestre)
- Local Hölder estimate (CI, Silvestre)
- Decay estimates for large velocities (CI, Mouhot, Silvestre)
- Schauder theory (CI, Silvestre)
- Global Hölder estimate and bootstrap (CI, Silvestre)

Why decay estimates?

- Boltzmann collision operator = non-local
- Control of coefficients $\boxed{\text{only if}}$ control of tails

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Rewriting Boltzmann (non-cutoff)

$$Q(f,f)(v) = \int_{\mathbb{R}^n} \int_{\mathbb{S}^{d-1}} \Big(f(v'_*)f(v') - f(v_*)f(v) \Big) B(|v - v_*|, \theta) \, d\sigma \, dv_*.$$

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$$+ \Big(f(v'_*)f(v) - f(v_*)f(v) \Big) B(|v-v_*|,\theta)\, d\sigma\, dv_*.$$

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$$Q(f,f)(v) = \int_{\mathbb{R}^n} \Big(f(v') - f(v) \Big) K_f(v'-v) \,\mathrm{d}\, v' + f(v)(c|\cdot|^\gamma * f).$$

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Boltzmann rewritten

$$(\partial_t + v \cdot \nabla_x) f = \underbrace{L_v f}_{} + \underbrace{h}_{}$$

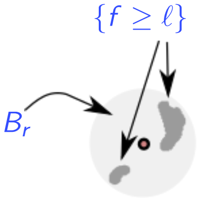
$$\text{with } L_v f = \int \Big(f(v') - f(v) \Big) K_f^{t,x,v}(v'-v) \,\mathrm{d}\, v'$$

Non-degeneracy from hydro controls

$$Q(f,f) = \int (f(v') - f(v)) K_f(v'-v)\, dv' + h$$

$$\boxed{K_f(v'-v) \simeq \frac{1}{|v'-v|^{d+2s}} \int_{w \perp v'-v} f(v+w)|w|^{\gamma+2s+1} dw}$$

Hydro controls yields $\{f \geq \ell\}$ has positive measure close to 0



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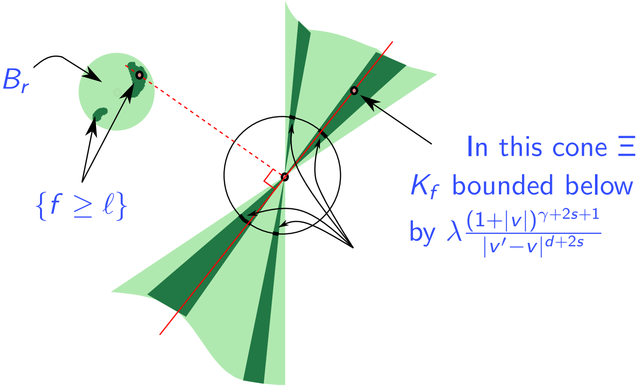

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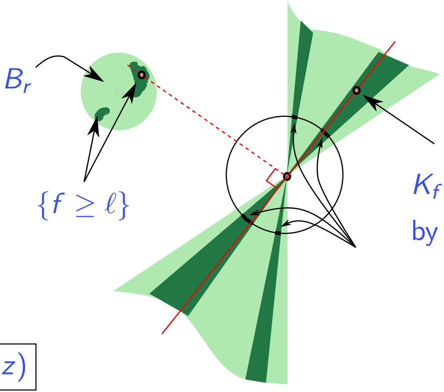
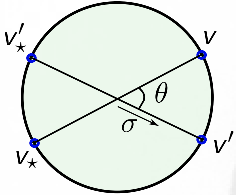


In this cone Ξ K_f bounded below by $\lambda \frac{(1+|v|)^{\gamma+2s+1}}{|v'-v|^{d+2s}}$

$$\boxed{K_f(z) = K_f(-z)}$$

The non-divergence form for Boltzmann



$$Q(f,f) = \iint (f_*' f' - f_* f) B dv_* d\sigma$$

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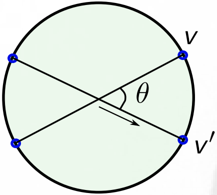

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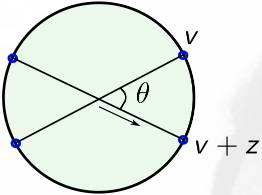

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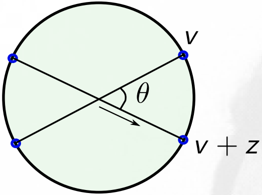

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Recall $\boxed{K_f(z) = K_f(-z)}$

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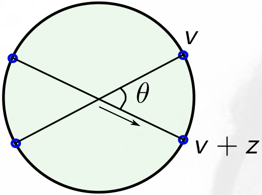

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Recall $\boxed{K_f(z) = K_f(-z)}$

$$= \frac{1}{2} \int (f(v+z) + f(v-z) - 2f(v)) K_f(v,z) dz + h$$

Barriers

$$Q(f,f) = \frac{1}{2}\int \Big(f(v+z) + f(v-z) - 2f(v) \Big) K_f(v,z)dz + h$$

See Silvestre '15, Schwab-Silvestre '16 (also CI-Silvestre '16)

Barriers

$$Q(f,f) = \frac{1}{2}\int \Big(f(v+z) + f(v-z) - 2f(v) \Big) K_f(v,z)dz + h$$

See Silvestre '15, Schwab-Silvestre '16 (also Cl-Silvestre '16)

- Important property: If f is smooth, then $Q(f,f)$ is bounded

$$Q(f,f) \leq \frac{1}{2}\|D^2 f\|_\infty \int_{|z|\leq 1} |z|^2 K_f(v,z)dz + \|f\|_\infty \int_{|z|\geq 1} K_f(v,z)dz + h$$

- Barriers: If $f \leq g$ with g smooth and $f(v_0) = g(v_0)$, then

$$Q(f,f) \leq \frac{1}{2}\int \Big(g(v+z) + g(v-z) - 2g(v) \Big) K_f(v,z)dz + h_g$$

at $v = v_0$

Maximum principle

Strategy

Goal: Prove $f(t,x,v) \leq \underbrace{N(t)(1+|v|)^{-q}}_{g(v)}$

Method: consider the barrier g and use the maximum principle

- Choose N and q such that $f \leq g$ at initial time
- Consider the first time t_* where $f \leq g$ fails

  $\Longrightarrow (\partial_t + v\nabla_x)f \geq (\partial_t + v\nabla_x)g$ at some (t_*, x_*, v_*)

  $\Longrightarrow N'(t_*)(1+|v_*|)^{-q} \leq Q(f,f)$

- contradict this inequality by using

  $$\left|\begin{array}{l} f \leq g \\ f(t_*, x_*, v_*) = g(t_*, x_*, v_*) \end{array}\right.$$

The good term

$$Q(f,f) \simeq \int \frac{f(v') - f(v)}{|v' - v|^{d+2s}} \left\{ \int_{w \perp v' - v} f(v+w) |w|^{\gamma+2s+1} dw \right\} dv + h$$

$$= \mathcal{G}(f,f) + (\text{bad terms}) + h$$

$$\mathcal{G}(f,f) \simeq \int_{|v+w| \leq c_q |v|} f(v+w)|w|^{\gamma+2s} \left\{ \int_{(v'-v)\perp w} (f(v') - f(v)) \frac{dv'}{|v'-v|^{d-1+2s}} \right\} dw$$

$\mathcal{G}(f,f)$ = The kernel adds $|v|^{\gamma+2s}$ & the operator differentiates $2s$ times

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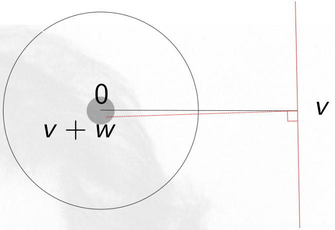
The good term

Pick $c_q \ll 1$

- Grey circle of radius $c_q|v|$
- Circle of radius $\frac{1}{2}|v|$



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$$\lesssim \int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s} \left\{ \int_{(v'-v)\perp w} (g(v') - g(v)) \frac{dv'}{|v'-v|^{d-1+2s}} \right\} dw$$

if $v' \simeq v$, the function g is concave & if not, use its decay

$$\lesssim \int_{|v+w|\le c_q|v|} f(v+w)|w|^{\gamma+2s} \left\{ \int_{\substack{(v'-v)\perp w \\ |v'-v|>r_q|v|}} (g(v') - g(v)) \frac{dv'}{|v'-v|^{d-1+2s}} \right\} dw$$

$$\lesssim -q^s|v|^{\gamma-q}$$

Recall the lower bound

- Cl-Mouhot-Silvestre. Decay estimates for large velocities in the Boltzmann equation without cut-off. *Journal de l'École Polytechnique — Mathématiques, Volume 7, p. 143-184 (2020)*

- CI-Mouhot-Silvestre. Decay estimates for large velocities in the Boltzmann equation without cut-off. *Journal de l'École Polytechnique — Mathématiques, Volume 7, p. 143-184 (2020)*

Thank you for your attention

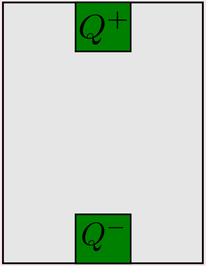
The Weak Harnack inequality

Theorem (Cl-Silvestre)

Let f be a non-negative supersolution *of the linear kinetic equation in* $(-1,0] \times B_1 \times B_1$. *Assume that the kernel satisfies the coercivity, upper bound, cancellation and non-degeneracy conditions. Then*

$$\boxed{\|f\|_{L^\varepsilon(Q^-)} \leq C \left(\inf_{Q^+} f + \|h\|_\infty \right)}$$

where ε and C only depends on dimension, λ and Λ.



Comments

- Weak Harnack implies Hölder

Weak Harnack inequality: comments

- Kinetic parabolic equations: $\partial_t f + v \cdot \nabla_x f = \nabla_v (A \nabla_v f)$
  - L^∞ bound: Pascucci, Polidoro (2004)
  - Hölder continuity: Wang, Zhang (2008, 2009, 2011)
  - Harnack inequality: Golse, CI, Mouhot, Vasseur (2015/2016)
- Fact: the full Harnack inequality may not hold true for integro-differential equations, see for instance Bogdan, Sztonyk (2005)
- Main ideas involved in the proof
  - Use ideas from **de Giorgi** (first lemma)
  - Transfer of regularity from v to x (**kinetic theory**)
  - Use ideas from **non-div form equations** (Krylov, Safonov)

Weak Harnack inequality: further comments

- First results for **kinetic** integro-differential equations

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- First results for **kinetic** integro-differential equations
- New regularity results for integro-differential equations in divergence form, *even in the homogeneous case*
    - **very mild assumptions on the kernel**
    - nonsymmetric kernels with cancellation conditions

Weak Harnack inequality: further comments

- First results for **kinetic** integro-differential equations
- New regularity results for integro-differential equations in divergence form, *even in the homogeneous case*
  - **very mild assumptions on the kernel**
  - nonsymmetric kernels with cancellation conditions
- Our results apply to the Boltzmann equation without cut-off
  - This is the reason why we are forced to work with so mild assumptions on the nonsymmetric kernel