Imperial College London

10 July 2022

Sparse spectral methods for fractional PDEs

PDE CDT Students & Alumni Reunion Event


John Papadopoulos[1]


Sheehan Olver[1]


José Carrillo[2]

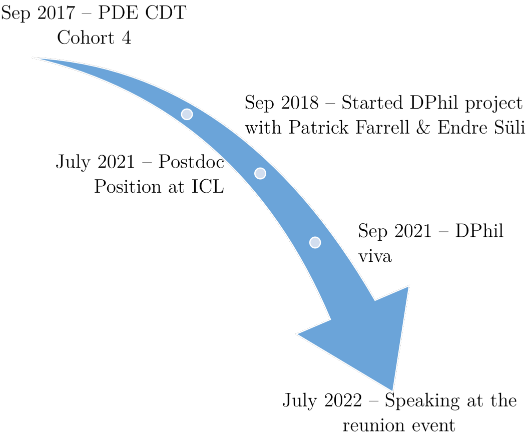

Timon Gutleb[2]


Bradley Treeby[3]

[1]Imperial College London; [2]University of Oxford; [3]UCL

My timeline

Imperial College London



My timeline

Imperial College London

MSc

- Mathematical modelling and scientific computing

PDE CDT

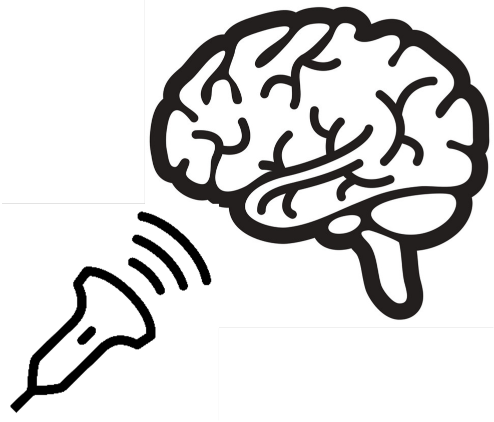
- Nonconvex optimisation problems with PDE constraints
- Existence, regularity & FEM analysis

Postdoc

- Fractional PDEs & spectral methods

Are fractional PDEs physical?

Imperial College London



FPDEs describe wave absorption in the brain[1].

[1] Images from https://clipart.world/brain-clipart/black-and-white-brain-clipart/, https://www.kindpng.com/imgv/iRoiRR_sound-wave-clipart-ultrasound-ultrasound-clip-art-hd/.

Other applications?

- Equilibrium measures;
- Dispersive transport of ions;
- Replacing total variation regularization in imaging;
- Geophysics problems with long range effects;
- Neural networks where all the layers are connected to every other layer.

Other applications?

Imperial College London

- Equilibrium measures;
- Dispersive transport of ions;
- Replacing total variation regularization in imaging;
- Geophysics problems with long range effects;
- Neural networks where all the layers are connected to every other layer.

Other applications?

Imperial College London

- Equilibrium measures;
- Dispersive transport of ions;
- Replacing total variation regularization in imaging;
- Geophysics problems with long range effects;
- Neural networks where all the layers are connected to every other layer.

Other applications?

Imperial College London

- Equilibrium measures;
- Dispersive transport of ions;
- Replacing total variation regularization in imaging;
- Geophysics problems with long range effects;
- Neural networks where all the layers are connected to every other layer.

Other applications?

Imperial College London

- Equilibrium measures;
- Dispersive transport of ions;
- Replacing total variation regularization in imaging;
- Geophysics problems with long range effects;
- Neural networks where all the layers are connected to every other layer.

The problem

Imperial College London

The PDE

Find $u \in H^{1/2}(\mathbb{R})$ that satisfies, for $\lambda \in \mathbb{R}$: $(\lambda\mathcal{I} + (-\Delta)^{1/2})u = f$.

$(-\Delta)^{1/2}$

Ten (or more) equivalent definitions of the fractional Laplacian over $\mathbb{R}^d$. E.g. for $s \in (0,1)$,

$$(-\Delta)^s u(x) := c_{d,s} \fint_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \mathrm{d}y$$

or

$$\mathcal{F}[(-\Delta)^s u](\omega) = |\omega|^{2s} \mathcal{F}[u](\omega).$$

The problem

Imperial College London

The PDE

Find $u \in H^{1/2}(\mathbb{R})$ that satisfies, for $\lambda \in \mathbb{R}$: $(\lambda \mathcal{I} + (-\Delta)^{1/2})u = f$.

$(-\Delta)^{1/2}$

Ten (or more) equivalent definitions of the fractional Laplacian over $\mathbb{R}^d$. E.g. for $s \in (0,1)$,

$$(-\Delta)^s u(x) := c_{d,s} \fint_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \, \mathrm{d}y$$

or

$$\mathcal{F}[(-\Delta)^s u](\omega) = |\omega|^{2s} \mathcal{F}[u](\omega).$$

The problem

Imperial College London

The PDE

Find $u \in H^{1/2}(\mathbb{R})$ that satisfies, for $\lambda \in \mathbb{R}$: $(\lambda\mathcal{I} + (-\Delta)^{1/2})u = f$.

$(-\Delta)^{1/2}$

Ten (or more) equivalent definitions of the fractional Laplacian over $\mathbb{R}^d$. E.g. for $s \in (0,1)$,

$$(-\Delta)^s u(x) \coloneqq c_{d,s} \fint_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \,\mathrm{d}y$$

or

$$\mathcal{F}[(-\Delta)^s u](\omega) = |\omega|^{2s} \mathcal{F}[u](\omega).$$

Fractional PDEs

Imperial College London

Observation

Solutions of fractional PDEs are “nonlocal” and can exhibit singularities.

Consequence

The solutions can be difficult to approximate numerically.

Challenge

How do we compute them with fast convergence?

Our proposal

A spectral method based on a so-called sum space.

Fractional PDEs

Imperial College London

Observation

Solutions of fractional PDEs are “nonlocal” and can exhibit singularities.

Consequence

The solutions can be difficult to approximate numerically.

Challenge

How do we compute them with fast convergence?

Our proposal

A spectral method based on a so-called sum space.

Fractional PDEs

Imperial College London

Observation

Solutions of fractional PDEs are “nonlocal” and can exhibit singularities.

Consequence

The solutions can be difficult to approximate numerically.

Challenge

How do we compute them with fast convergence?

Our proposal

A spectral method based on a so-called sum space.

Fractional PDEs

Imperial College London

Observation

Solutions of fractional PDEs are “nonlocal” and can exhibit singularities.

Consequence

The solutions can be difficult to approximate numerically.

Challenge

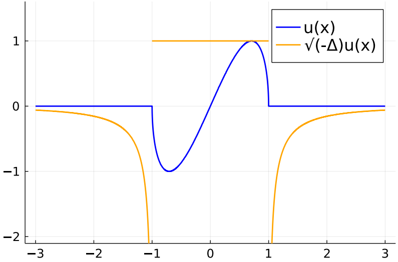
How do we compute them with fast convergence?

Our proposal

A spectral method based on a so-called sum space.

Singularities and non-locality

The fractional Laplacian is not local. E.g.



Nonlocal

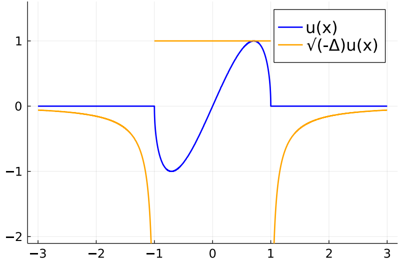
$u(x) = 0$ for $|x| \geq 1$ but $(-\Delta)^{1/2}u(x) \neq 0$ for all $x \in \mathbb{R}$.

Singularities

As $x \downarrow 1$ and $x \uparrow -1$, then $|(-\Delta)^{1/2}u(x)| \to \infty$.

Singularities and non-locality

The fractional Laplacian is not local. E.g.



Nonlocal

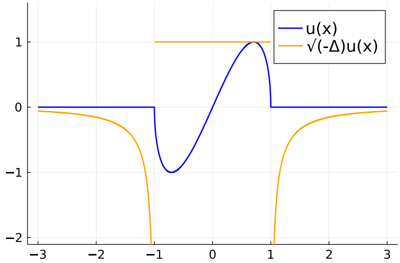
$u(x) = 0$ for $|x| \geq 1$ but $(-\Delta)^{1/2}u(x) \neq 0$ for all $x \in \mathbb{R}$.

Singularities

As $x \downarrow 1$ and $x \uparrow -1$, then $|(-\Delta)^{1/2}u(x)| \to \infty$.

Singularities and non-locality

The fractional Laplacian is not local. E.g.



Nonlocal

$u(x) = 0$ for $|x| \geq 1$ but $(-\Delta)^{1/2}u(x) \neq 0$ for all $x \in \mathbb{R}$.

Singularities

As $x \downarrow 1$ and $x \uparrow -1$, then $|(-\Delta)^{1/2}u(x)| \to \infty$.

Spectral methods

Imperial College London

Pick an orthogonal set of polynomials, e.g. the so-called *ChebyshevT* polynomials, denoted $\{T_n(x)\}$. These satisfy

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \delta_{nm};\ T_0(x) = 1,\ \ T_1(x) = x,\ \ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

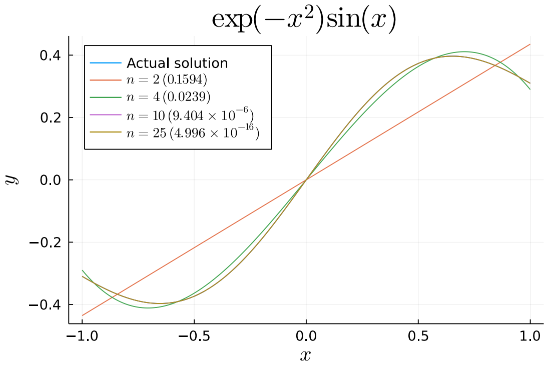
For $x \in [-1, 1]$, consider the approximation: $\mathrm{e}^{-x^2}\sin(x) \approx \sum_{k=0}^{n} c_k\, T_k(x)$.

Spectral methods

Pick an orthogonal set of polynomials, e.g. the so-called *ChebyshevT* polynomials, denoted $\{T_n(x)\}$. These satisfy

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \delta_{nm};\ T_0(x) = 1,\ \ T_1(x) = x,\ \ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

For $x \in [-1, 1]$, consider the approximation: $\mathrm{e}^{-x^2}\sin(x) \approx \sum_{k=0}^{n} c_k\,T_k(x)$.



Sparse spectral methods

Imperial College London

Many spectral methods for differential equations induce *dense* matrices ❌. Consider solving

$$-u'(x) = f(x), \quad u(-1) = 0.$$

A spectral method recipe 📝:

1. Expand $f(x)$ in the ChebyshevT polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via a collocation method. D is **dense**.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

Imperial College London

Many spectral methods for differential equations induce *dense* matrices ❌. Consider solving

$$-u'(x) = f(x), \quad u(-1) = 0.$$

A spectral method recipe 📝:

1. Expand $f(x)$ in the ChebyshevT polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via a collocation method. D is **dense**.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

Many spectral methods for differential equations induce *dense* matrices ✗. Consider solving

$$-u'(x) = f(x), \;\; u(-1) = 0.$$

A spectral method recipe:

1. Expand $f(x)$ in the ChebyshevT polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via a collocation method. D is **dense**.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

Imperial College London

Many spectral methods for differential equations induce *dense* matrices ❌. Consider solving

$$-u'(x) = f(x), \quad u(-1) = 0.$$

A spectral method recipe 📝:

1. Expand $f(x)$ in the ChebyshevT polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via a collocation method. D is **dense**.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

Imperial College London

Many spectral methods for differential equations induce *dense* matrices ❌. Consider solving

$$-u'(x) = f(x), \;\; u(-1) = 0.$$

A spectral method recipe 📝:

1. Expand $f(x)$ in the ChebyshevT polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via a collocation method. D is **dense**.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

An observation

Let $\{U_n\}$ denote the *ChebyshevU* orthogonal polynomial basis. Then, for $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* form:

$$\left(T_0'(x)\ T_1'(x)\ T_2'(x)\ \ldots\right) \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} = \left(U_0(x)\ U_1(x)\ U_2(x)\ \ldots\right)$$

A sparse spectral method recipe:

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via ChebyshevT/U relationship. D is sparse with one dense row related to the BC.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

An observation

Let $\{U_n\}$ denote the *ChebyshevU* orthogonal polynomial basis. Then, for $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* form:

$$\left(T_0'(x)\ T_1'(x)\ T_2'(x)\ \ldots\right) \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} = \left(U_0(x)\ U_1(x)\ U_2(x)\ \ldots\right)$$

A sparse spectral method recipe:

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via ChebyshevT/U relationship. D is sparse with one dense row related to the BC.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

Imperial College London

An observation

Let $\{U_n\}$ denote the *ChebyshevU* orthogonal polynomial basis. Then, for $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* form:

$$\left(T_0'(x)\ T_1'(x)\ T_2'(x)\ \ldots\right)\begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} = \left(U_0(x)\ U_1(x)\ U_2(x)\ \ldots\right)$$

A sparse spectral method recipe:

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via ChebyshevT/U relationship. D is sparse with one dense row related to the BC.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

An observation

Let $\{U_n\}$ denote the *ChebyshevU* orthogonal polynomial basis. Then, for $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* form:

$$\begin{pmatrix} T_0'(x) & T_1'(x) & T_2'(x) & \dots \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} U_0(x) & U_1(x) & U_2(x) & \dots \end{pmatrix}$$

A sparse spectral method recipe:

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via ChebyshevT/U relationship. D is sparse with one dense row related to the BC.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

An observation

Let $\{U_n\}$ denote the *ChebyshevU* orthogonal polynomial basis. Then, for $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* form:

$$\begin{pmatrix} T_0'(x) & T_1'(x) & T_2'(x) & \dots \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix} = \begin{pmatrix} U_0(x) & U_1(x) & U_2(x) & \dots \end{pmatrix}$$

A sparse spectral method recipe:

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via ChebyshevT/U relationship. D is sparse with one dense row related to the BC.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral methods

Imperial College London

An observation

Let $\{U_n\}$ denote the *ChebyshevU* orthogonal polynomial basis. Then, for $n \geq 1$, $T_n'(x) = nU_{n-1}(x)$. Or in *quasimatrix* form:

$$\left(T_0'(x)\ T_1'(x)\ T_2'(x)\ \ldots\right)\begin{pmatrix}0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots\end{pmatrix} = \left(U_0(x)\ U_1(x)\ U_2(x)\ \ldots\right)$$

A sparse spectral method recipe:

1. Expand $f(x)$ in the ChebyshevU polynomial basis, truncate, and collect the coefficients in vector $\mathbf{f}$.
2. Construct the differential operator D via ChebyshevT/U relationship. D is sparse with one dense row related to the BC.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the ChebyshevT expansion of $u(x)$.

Sparse spectral method for FPDE

Imperial College London

ChebyshevT

$T_n(x) =$ ChebyshevT, polynomials of order n, $x \in [-1, 1]$, orthogonal w.r.t. $1/\sqrt{1 - x^2}$. Extend by 0 to $\mathbb{R}$ outside $[-1, 1]$.

ChebyshevU

$U_n(x) =$ ChebyshevU, polynomials of order n, $x \in [-1, 1]$, orthogonal w.r.t. $\sqrt{1 - x^2}$. Extend by 0 to $\mathbb{R}$ outside $[-1, 1]$.

Extended Chebyshev functions on $\mathbb{R}$

$\tilde{T}_n(x)$ and $\tilde{U}_n(x)$ are carefully chosen extensions of $T_n(x)$ and $U_n(x)$, respectively, for $x \in \mathbb{R}$.

Sparse spectral method for FPDE

ChebyshevT

$T_n(x)$ = ChebyshevT, polynomials of order n, $x \in [-1, 1]$, orthogonal w.r.t. $1/\sqrt{1-x^2}$. Extend by 0 to $\mathbb{R}$ outside $[-1, 1]$.

ChebyshevU

$U_n(x)$ = ChebyshevU, polynomials of order n, $x \in [-1, 1]$, orthogonal w.r.t. $\sqrt{1-x^2}$. Extend by 0 to $\mathbb{R}$ outside $[-1, 1]$.

Extended Chebyshev functions on $\mathbb{R}$

$\tilde{T}_n(x)$ and $\tilde{U}_n(x)$ are carefully chosen extensions of $T_n(x)$ and $U_n(x)$, respectively, for $x \in \mathbb{R}$.

Sparse spectral method for FPDE

Imperial College London

ChebyshevT

$T_n(x) =$ ChebyshevT, polynomials of order n, $x \in [-1, 1]$, orthogonal w.r.t. $1/\sqrt{1-x^2}$. Extend by 0 to $\mathbb{R}$ outside $[-1, 1]$.

ChebyshevU

$U_n(x) =$ ChebyshevU, polynomials of order n, $x \in [-1, 1]$, orthogonal w.r.t. $\sqrt{1-x^2}$. Extend by 0 to $\mathbb{R}$ outside $[-1, 1]$.

Extended Chebyshev functions on $\mathbb{R}$

$\tilde{T}_n(x)$ and $\tilde{U}_n(x)$ are carefully chosen extensions of $T_n(x)$ and $U_n(x)$, respectively, for $x \in \mathbb{R}$.

Extended Chebyshev functions

Imperial College London

$$\tilde{T}_n(x) := \begin{cases} T_n(x) & |x| \leq 1, \\ (x - \operatorname{sgn}(x)\sqrt{x^2 - 1})^n & |x| > 1. \end{cases}$$

$$\tilde{U}_n(x) := \tilde{T}_n(x) - \tilde{U}_{n-2}(x), \quad n \geq 0,$$

where

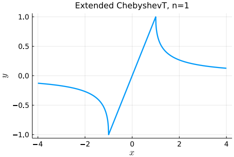
$$\hat{U}_{-1}(x) := \begin{cases} 0 & |x| \leq 1, \\ -\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}} & |x| > 1, \end{cases}$$

$$\hat{U}_{-2}(x) := x\hat{U}_{-1}(x).$$

Extended Chebyshev functions

Imperial College London

$$\tilde{T}_n(x) := \begin{cases} T_n(x) & |x| \leq 1, \\ (x - \operatorname{sgn}(x)\sqrt{x^2 - 1})^n & |x| > 1. \end{cases}$$



$$\tilde{U}_n(x) := \tilde{T}_n(x) - \tilde{U}_{n-2}(x), \quad n \geq 0,$$

where

$$\hat{U}_{-1}(x) := \begin{cases} 0 & |x| \leq 1, \\ -\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}} & |x| > 1, \end{cases}$$

$$\hat{U}_{-2}(x) := x\hat{U}_{-1}(x).$$

Extended Chebyshev functions

Imperial College London

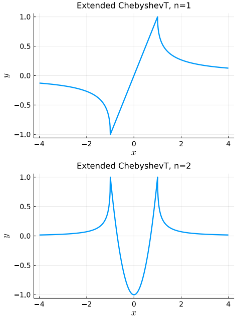
$$\tilde{T}_n(x) := \begin{cases} T_n(x) & |x| \le 1, \\ (x - \operatorname{sgn}(x)\sqrt{x^2-1})^n & |x| > 1. \end{cases}$$

$$\tilde{U}_n(x) := \tilde{T}_n(x) - \tilde{U}_{n-2}(x), \quad n \ge 0,$$

where

$$\hat{U}_{-1}(x) := \begin{cases} 0 & |x| \le 1, \\ -\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}} & |x| > 1, \end{cases}$$

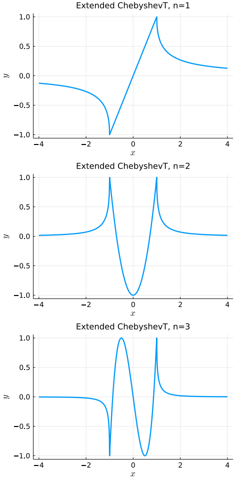
$$\hat{U}_{-2}(x) := x\hat{U}_{-1}(x).$$



Extended Chebyshev functions

Imperial College London

$$\tilde{T}_n(x) := \begin{cases} T_n(x) & |x| \leq 1, \\ (x - \mathrm{sgn}(x)\sqrt{x^2 - 1})^n & |x| > 1. \end{cases}$$



$$\tilde{U}_n(x) := \tilde{T}_n(x) - \tilde{U}_{n-2}(x), \quad n \geq 0,$$

where

$$\hat{U}_{-1}(x) := \begin{cases} 0 & |x| \leq 1, \\ -\frac{\mathrm{sgn}(x)}{\sqrt{x^2-1}} & |x| > 1, \end{cases}$$

$$\hat{U}_{-2}(x) := x\hat{U}_{-1}(x).$$

Extended Chebyshev functions

Imperial College London

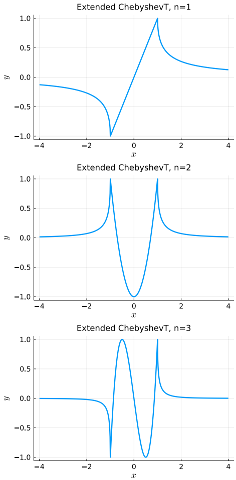
$$\tilde{T}_n(x) := \begin{cases} T_n(x) & |x| \le 1, \\ (x - \operatorname{sgn}(x)\sqrt{x^2-1})^n & |x| > 1. \end{cases}$$

$$\tilde{U}_n(x) := \tilde{T}_n(x) - \tilde{U}_{n-2}(x), \quad n \ge 0,$$

where

$$\hat{U}_{-1}(x) := \begin{cases} 0 & |x| \le 1, \\ -\frac{\operatorname{sgn}(x)}{\sqrt{x^2-1}} & |x| > 1, \end{cases}$$

$$\hat{U}_{-2}(x) := x\hat{U}_{-1}(x).$$



Motivation for choice of functions

Imperial College London

Denote the Hilbert transform: $\mathcal{H}[u](x) \coloneqq \frac{1}{\pi} \fint_{\mathbb{R}} \frac{u(y)}{x-y} \mathrm{d}y$. Let $W_n(x) \coloneqq (1-x^2)_+^{1/2} U_n(x)$.

Property 1

- $\mathcal{H}[W_n](x) = \tilde{T}_{n+1}(x)$ for all $x \in \mathbb{R}$.
- $\mathcal{H}[\tilde{T}_{n+1}](x) = -W_n(x)$ for all $x \in \mathbb{R}$.

Property 2

Modulo some assumptions: $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{H} = (-\Delta)^{1/2}$ in 1D.

Property 3

- $\frac{\mathrm{d}}{\mathrm{d}x}[W_n](x) = (n+1)(1-x^2)_+^{-1/2} T_{n+1}(x)$.
- $\frac{\mathrm{d}}{\mathrm{d}x}[\tilde{T}_{n+1}](x) = n\tilde{U}_{n-1}(x)$.

Motivation for choice of functions

Imperial College London

Denote the Hilbert transform: $\mathcal{H}[u](x) \coloneqq \frac{1}{\pi} \fint_{\mathbb{R}} \frac{u(y)}{x-y} \mathrm{d}y$. Let $W_n(x) \coloneqq (1-x^2)_+^{1/2} U_n(x)$.

Property 1

- $\mathcal{H}[W_n](x) = \tilde{T}_{n+1}(x)$ for all $x \in \mathbb{R}$.
- $\mathcal{H}[\tilde{T}_{n+1}](x) = -W_n(x)$ for all $x \in \mathbb{R}$.

Property 2

Modulo some assumptions: $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{H} = (-\Delta)^{1/2}$ in 1D.

Property 3

- $\frac{\mathrm{d}}{\mathrm{d}x}[W_n](x) = (n+1)(1-x^2)_+^{-1/2} T_{n+1}(x)$.
- $\frac{\mathrm{d}}{\mathrm{d}x}[\tilde{T}_{n+1}](x) = n\tilde{U}_{n-1}(x)$.

Motivation for choice of functions

Imperial College London

Denote the Hilbert transform: $\mathcal{H}[u](x) := \frac{1}{\pi} ⨍_{\mathbb{R}} \frac{u(y)}{x-y} \mathrm{d}y$. Let $W_n(x) := (1-x^2)_+^{1/2} U_n(x)$.

Property 1

- $\mathcal{H}[W_n](x) = \tilde{T}_{n+1}(x)$ for all $x \in \mathbb{R}$.
- $\mathcal{H}[\tilde{T}_{n+1}](x) = -W_n(x)$ for all $x \in \mathbb{R}$.

Property 2

Modulo some assumptions: $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{H} = (-\Delta)^{1/2}$ in 1D.

Property 3

- $\frac{\mathrm{d}}{\mathrm{d}x}[W_n](x) = (n+1)(1-x^2)_+^{-1/2} T_{n+1}(x)$.
- $\frac{\mathrm{d}}{\mathrm{d}x}[\tilde{T}_{n+1}](x) = n\tilde{U}_{n-1}(x)$.

Motivation for choice of functions

Imperial College London

Denote the Hilbert transform: $\mathcal{H}[u](x) \coloneqq \frac{1}{\pi} ⨍_{\mathbb{R}} \frac{u(y)}{x-y} \mathrm{d}y$. Let $W_n(x) \coloneqq (1-x^2)_+^{1/2} U_n(x)$.

Property 1

- $\mathcal{H}[W_n](x) = \tilde{T}_{n+1}(x)$ for all $x \in \mathbb{R}$.
- $\mathcal{H}[\tilde{T}_{n+1}](x) = -W_n(x)$ for all $x \in \mathbb{R}$.

Property 2

Modulo some assumptions: $\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{H} = (-\Delta)^{1/2}$ in 1D.

Property 3

- $\frac{\mathrm{d}}{\mathrm{d}x}[W_n](x) = (n+1)(1-x^2)_+^{-1/2} T_{n+1}(x)$.
- $\frac{\mathrm{d}}{\mathrm{d}x}[\tilde{T}_{n+1}](x) = n\tilde{U}_{n-1}(x)$.

A sparse spectral method for an FPDE

$(-\Delta)^{1/2}$

$$\{(1-x^2)_+^{1/2} U_n(x)\} \xrightarrow{(-\Delta)^{1/2}} \{\tilde{U}_n(x)\},$$

$$\{\tilde{T}_n(x)\} \xrightarrow{(-\Delta)^{1/2}} \{(1-x^2)_+^{-1/2} T_n(x)\}.$$

Identity

$$\{(1-x^2)_+^{1/2} U_n(x)\} \xrightarrow{\mathcal{I}} \{(1-x^2)_+^{-1/2} T_n(x)\},$$

$$\{\tilde{T}_n(x)\} \xrightarrow{\mathcal{I}} \{\tilde{U}_n(x)\}.$$

A sparse spectral method for an FPDE

Key idea: use sum space $\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}$.

$\lambda \mathcal{I} + (-\Delta)^{1/2}$

$$\underbrace{\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}}_{\text{sum space}} \xrightarrow{\lambda\mathcal{I}+(-\Delta)^{1/2}} \underbrace{\{\tilde{U}_n(x)\} \cup \{(1-x^2)_+^{-1/2} T_n(x)\}}_{\text{dual sum space}}.$$

A sparse spectral method recipe:

1. Expand f in the dual sum space, truncate, and collect coefficients in vector $\mathbf{f}$.
2. Construct the truncated **sparse** matrix D induced by $(\lambda\mathcal{I} + (-\Delta)^{1/2})$.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the sum space expansion of $u(x)$, $x \in \mathbb{R}$.

A sparse spectral method for an FPDE

Imperial College London

Key idea: use sum space $\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}$.

$\lambda\mathcal{I} + (-\Delta)^{1/2}$

$$\underbrace{\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}}_{\text{sum space}} \xrightarrow{\lambda\mathcal{I}+(-\Delta)^{1/2}} \underbrace{\{\tilde{U}_n(x)\} \cup \{(1-x^2)_+^{-1/2} T_n(x)\}}_{\text{dual sum space}}.$$

A sparse spectral method recipe 📝:

1. Expand f in the dual sum space, truncate, and collect coefficients in vector $\mathbf{f}$.
2. Construct the truncated **sparse** matrix D induced by $(\lambda\mathcal{I} + (-\Delta)^{1/2})$.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the sum space expansion of $u(x)$, $x \in \mathbb{R}$.

A sparse spectral method for an FPDE

Imperial College London

Key idea: use sum space $\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}$.

$\lambda\mathcal{I} + (-\Delta)^{1/2}$

$$\underbrace{\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}}_{\text{sum space}} \xrightarrow{\lambda\mathcal{I}+(-\Delta)^{1/2}} \underbrace{\{\tilde{U}_n(x)\} \cup \{(1-x^2)_+^{-1/2} T_n(x)\}}_{\text{dual sum space}}.$$

A sparse spectral method recipe:

1. Expand f in the dual sum space, truncate, and collect coefficients in vector $\mathbf{f}$.
2. Construct the truncated sparse matrix D induced by $(\lambda\mathcal{I} + (-\Delta)^{1/2})$.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the sum space expansion of $u(x)$, $x \in \mathbb{R}$.

A sparse spectral method for an FPDE

Key idea: use sum space $\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}$.

$\lambda\mathcal{I} + (-\Delta)^{1/2}$

$$\underbrace{\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}}_{\text{sum space}} \xrightarrow{\lambda\mathcal{I}+(-\Delta)^{1/2}} \underbrace{\{\tilde{U}_n(x)\} \cup \{(1-x^2)_+^{-1/2} T_n(x)\}}_{\text{dual sum space}}.$$

A sparse spectral method recipe:

1. Expand f in the dual sum space, truncate, and collect coefficients in vector $\mathbf{f}$.
2. Construct the truncated **sparse** matrix D induced by $(\lambda\mathcal{I} + (-\Delta)^{1/2})$.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the sum space expansion of $u(x)$, $x \in \mathbb{R}$.

A sparse spectral method for an FPDE

Imperial College London

Key idea: use sum space $\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}$.

$\lambda \mathcal{I} + (-\Delta)^{1/2}$

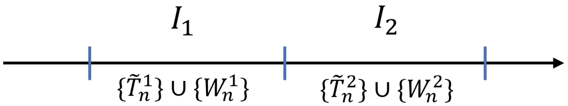
$$\underbrace{\{\tilde{T}_n(x)\} \cup \{(1-x^2)_+^{1/2} U_n(x)\}}_{\text{sum space}} \xrightarrow{\lambda \mathcal{I} + (-\Delta)^{1/2}} \underbrace{\{\tilde{U}_n(x)\} \cup \{(1-x^2)_+^{-1/2} T_n(x)\}}_{\text{dual sum space}}.$$

A sparse spectral method recipe 📝:

1. Expand f in the dual sum space, truncate, and collect coefficients in vector $\mathbf{f}$.
2. Construct the truncated **sparse** matrix D induced by $(\lambda \mathcal{I} + (-\Delta)^{1/2})$.
3. Solve $D\mathbf{u} = \mathbf{f}$ for the coefficients $\mathbf{u}$ in the sum space expansion of $u(x)$, $x \in \mathbb{R}$.

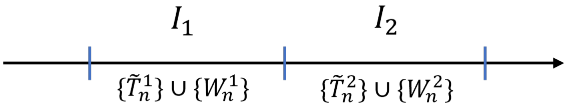
Combining different intervals

Imperial College London



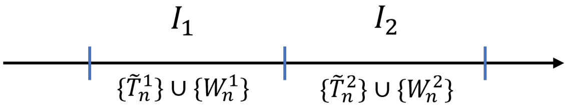
- $\{\tilde{T}_n^i\}$ and $\{W_n^i\}$, $i = 1, 2$, are the spaces of affine transformed functions of the spaces $\{\tilde{T}_n\}$ and $\{\sqrt{1-x^2}U_n\}$, respectively, centred at the interval midpoints of I_1 and I_2.
- W_n^i are supported on I_i, $i = 1, 2$.
- $\tilde{T}_n^i$ are not only supported on I_i and interact with other functions centred on other intervals.

Combining different intervals



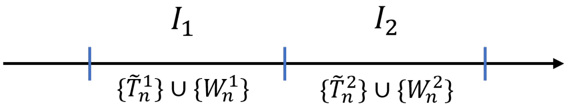
- $\{\tilde{T}_n^i\}$ and $\{W_n^i\}$, $i = 1, 2$, are the spaces of affine transformed functions of the spaces $\{\tilde{T}_n\}$ and $\{\sqrt{1-x^2}U_n\}$, respectively, centred at the interval midpoints of I_1 and I_2.
- W_n^i are supported on I_i, $i = 1, 2$.
- $\tilde{T}_n^i$ are not only supported on I_i and interact with other functions centred on other intervals.

Combining different intervals



- $\{\tilde{T}_n^i\}$ and $\{W_n^i\}$, $i = 1, 2$, are the spaces of affine transformed functions of the spaces $\{\tilde{T}_n\}$ and $\{\sqrt{1-x^2}U_n\}$, respectively, centred at the interval midpoints of I_1 and I_2.
- W_n^i are supported on I_i, $i = 1, 2$.
- $\tilde{T}_n^i$ are not only supported on I_i and interact with other functions centred on other intervals.

Combining different intervals



$\lambda\mathcal{I} + (-\Delta)^{1/2}$ induces a block-diagonal matrix. Each block corresponds to the sum space centred at one interval.

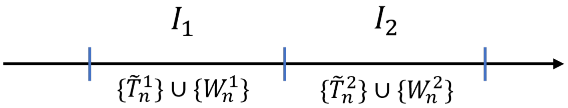
$$\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}.$$

This decomposes to

$$D_1\mathbf{u}_1 = \mathbf{f}_1, \quad D_2\mathbf{u}_2 = \mathbf{f}_2.$$

Combining different intervals

Imperial College London



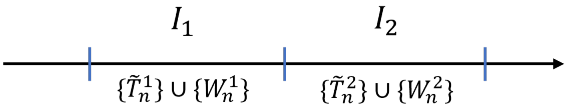
$\lambda\mathcal{I} + (-\Delta)^{1/2}$ induces a block-diagonal matrix. Each block corresponds to the sum space centred at one interval.

$$\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}.$$

This decomposes to

$$D_1\mathbf{u}_1 = \mathbf{f}_1, \quad D_2\mathbf{u}_2 = \mathbf{f}_2.$$

Combining different intervals



$\lambda\mathcal{I} + (-\Delta)^{1/2}$ induces a block-diagonal matrix. Each block corresponds to the sum space centred at one interval.

$$\begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}.$$

This decomposes to

$$D_1 \mathbf{u}_1 = \mathbf{f}_1, \quad D_2 \mathbf{u}_2 = \mathbf{f}_2.$$

Example: the Gaussian

Imperial College London

$$(\mathcal{I} + (-\Delta)^{1/2})u(x) = \mathrm{e}^{-x^2} + \frac{2}{\sqrt{\pi}} {}_1F_1(1; 1/2; -x^2).$$

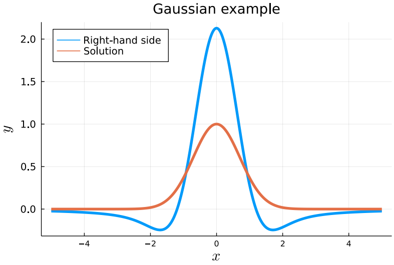
${}_1F_1$ is the Kummer confluent hypergeometric function. The solution is $u(x) = \mathrm{e}^{-x^2}$.

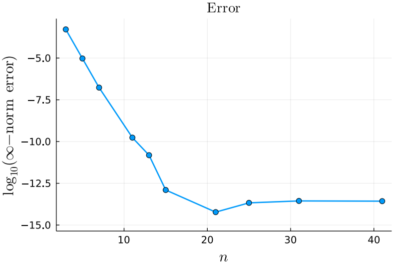
Example: the Gaussian

Imperial College London

$$(\mathcal{I} + (-\Delta)^{1/2})u(x) = \mathrm{e}^{-x^2} + \frac{2}{\sqrt{\pi}} {}_1F_1(1; 1/2; -x^2).$$

${}_1F_1$ is the Kummer confluent hypergeometric function. The solution is $u(x) = \mathrm{e}^{-x^2}$.





Example: wave propagation

Consider the FPDE ($u(x,t) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} + \frac{\partial^2}{\partial t^2}]u(x,t) = (1-x^2)_+^{1/2} U_4(x) \mathrm{e}^{-t^2}.$$

A Fourier transform in time gives ($\hat{u}(x,\omega) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} - \omega^2]\hat{u}(x,\omega) = \sqrt{\pi}(1-x^2)_+^{1/2} U_4(x) \mathrm{e}^{-\omega^2/4}.$$

Example: wave propagation

Imperial College London

Consider the FPDE ($u(x, t) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} + \frac{\partial^2}{\partial t^2}]u(x, t) = (1 - x^2)_+^{1/2} U_4(x)\mathrm{e}^{-t^2}.$$

A Fourier transform in time gives ($\hat{u}(x, \omega) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} - \omega^2]\hat{u}(x, \omega) = \sqrt{\pi}(1 - x^2)_+^{1/2} U_4(x)\mathrm{e}^{-\omega^2/4}.$$

Example: wave propagation

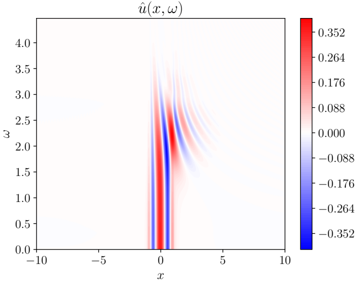
Imperial College London

Consider the FPDE ($u(x,t) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} + \frac{\partial^2}{\partial t^2}]u(x,t) = (1-x^2)_+^{1/2} U_4(x)\mathrm{e}^{-t^2}.$$

A Fourier transform in time gives ($\hat{u}(x,\omega) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} - \omega^2]\hat{u}(x,\omega) = \sqrt{\pi}(1-x^2)_+^{1/2} U_4(x)\mathrm{e}^{-\omega^2/4}.$$



Example: wave propagation

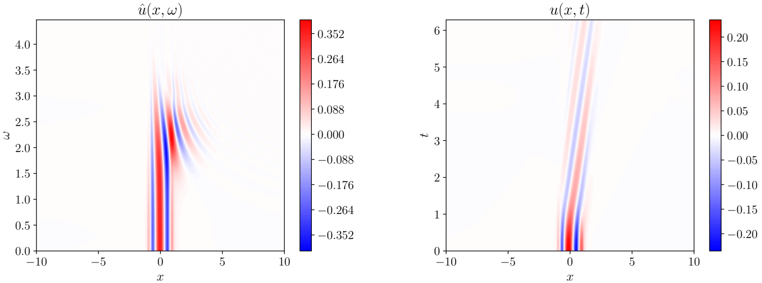
Imperial College London

Consider the FPDE ($u(x,t) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} + \frac{\partial^2}{\partial t^2}]u(x,t) = (1-x^2)_+^{1/2} U_4(x) \mathrm{e}^{-t^2}.$$

A Fourier transform in time gives ($\hat{u}(x,\omega) \to 0$ as $|x| \to \infty$):

$$[(-\Delta)^{1/2} + \mathcal{H} - \omega^2]\hat{u}(x,\omega) = \sqrt{\pi}(1-x^2)_+^{1/2} U_4(x) \mathrm{e}^{-\omega^2/4}.$$



Conclusions

Imperial College London

- A sparse spectral method for solving the identity + half-Laplacian;
- Based on a carefully chosen sum space.

Still to come

- An efficient implementation written in `Julia` see `https://github.com/ioannisPApapadopoulos/SumSpaces.jl`;
- Generalization to $(-\Delta)^s$ with $s \in (0, 1)$;
- Generalization to higher dimensions.

Conclusions

Imperial College London

- A sparse spectral method for solving the identity + half-Laplacian;
- Based on a carefully chosen sum space.

Still to come

- An efficient implementation written in `Julia` see `https://github.com/ioannisPApapadopoulos/SumSpaces.jl`;
- Generalization to $(-\Delta)^s$ with $s \in (0, 1)$;
- Generalization to higher dimensions.

Thank you for listening!

✉ ioannis.papadopoulos13@imperial.ac.uk