# Global regularity estimates for the homogeneous Landau equation.

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# Collisional kinetic equations

The Landau equation is a kinetic equation used to model motion of particles in plasma. In kinetic equations, the behavior of a collection of particles is described in terms of a statistical quantity, called the *particle density* $f(x, v, t)$.

- Averages of $f$ in the velocity variable $v$ define observable quantities:

$$
\begin{aligned}
\rho(x,t) &= \int f(x,v,t)\,dv,\\
\rho(x,t)u(x,t) &= \int v\,f(x,v,t)\,dv,\\
\rho(x,t)u^2(x,t) + E(x,t) &= \int |v|^2 f(x,v,t)\,dv.
\end{aligned}
$$

# The inhomogeneous Landau equation

The Landau equation (1936) reads as

$$\partial_t f + v \cdot \nabla_x f = Q_L(f,f)$$

with $f(x,v,t)$ and

$$Q_L(f,f) := \operatorname{div}_v \int_{\mathbb{R}^3} \frac{\mathbb{P}(w-v)}{|w-v|}(f(w)\nabla_v f(v) - f(v)\nabla_w f(w))\, dw$$

$$\mathbb{P}(z) := \left(\mathbb{I} - \frac{z \otimes z}{|z|^2}\right)$$

$Q_L(f,f)$ describes how particles collide:

- $Q_L(f,f)$ only acts in the velocity variable (local in $x$ and $t$).
- $Q_L(f,f)$ is quadratic in $f$: collisions of three or more particles are neglected.
- For all $x$ and $t > 0$ we have **3 conservation laws** and **decay of entropy functional**

$$\int Q_L(f,f)(1,v,|v|^2)\, dv = 0, \quad \int Q_L(f,f)\, ln\, f\, dv \leq 0$$

# The homogeneous Landau equation

The **homogeneous** Landau equation reads as

$$\partial_t f = Q_L(f, f),$$

with $f = f(v, t)$ and

$$Q_L(f, f) = \mathrm{div} \int_{\mathbb{R}^3} \frac{\mathbb{P}(w - v)}{|w - v|} (f(w) \nabla_v f(v) - f(v) \nabla_w f(w)) \, dw$$

- The 3 conservation laws and decay of entropy functional imply

$$\int f(1, v, |v|^2) \, dv = constant, \quad \partial_t \int f \; ln \, f \; dv \leq 0$$

Main goal: **understanding global regularity** versus **blow-up in finite time**.

# Global well-posedness versus blow-up in finite time

The homogeneous Landau equation is a **quasi-linear parabolic** equation with **quadratic non-linearity** and **anisotropic** diffusion:

$$Q_L(f,f) = \mathrm{div} \int_{\mathbb{R}^3} \frac{\mathbb{P}(w-v)}{|w-v|}(f(w)\nabla_v f(v) - f(v)\nabla_w f(w))\, dw$$
$$= \mathrm{div}(A[f]\nabla f - f\nabla a[f])$$
$$= Tr(A[f]Hess(f)) + f^2$$

where

$$A[f](v,t) := \int_{\mathbb{R}^3} \frac{\mathbb{P}(w-v)}{|w-v|} f(w)\, dw, \quad a[f](v,t) := 2\int_{\mathbb{R}^3} \frac{1}{|w-v|} f(w)\, dw,$$

The equation is parabolic, thanks to the conservation of mass, first and second momentum:

$$\frac{C}{1+|v|^3} \le A[f] \le a[f] \qquad \forall\ t > 0.$$

To bound $A[f], a[f]$ and $\nabla a[f]$ from above we need more integrability for $f$:

$$A[f], a[f] \le C(\|f\|_{L^\infty(0,T,L^1(\mathbb{R}^3))}, \|f\|_{L^\infty(0,T,L^p(\mathbb{R}^3))}), \quad p > 3/2$$

# The $L^{3/2^+}$ criteria

## Theorem

*Let $f(v,t)$ be a smooth solution to the homogeneous Landau equation with initial data such that*

$$\int_{\mathbb{R}^3} f_{in}(1, v, |v|^2, ln(f_{in}))\, dv < +\infty.$$

*Assume that $[0,T]$ is a smooth solution's maximal interval of existence. Then*

$$\|f\|_{L^\infty(0,t,L^{3/2^+}(\mathbb{R}^3))} \xrightarrow{t\to T} +\infty.$$

# The $L^{3/2^+}$ criteria: proof

## Proof.

Assume $\|f\|_{L^\infty(0,T,L^{3/2^+}\cap L^1(\mathbb{R}^3))} \leq C$. There exists a constant $\varepsilon$ such that

$$\int_{\mathbb{R}^3} f\phi^2 \, dv \leq \varepsilon \int_{\mathbb{R}^3} \langle A[f]\nabla\phi, \nabla\phi\rangle \, dv + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \phi^2 \, dv, \quad \forall \phi \in C_0^\infty.$$

Perform now $L^p$ estimate on the Landau equation:

$$\partial_t \int_{\mathbb{R}^3} f^p \, dv = -\int_{\mathbb{R}^3} \langle A[f]\nabla f^{\frac{p}{2}}, \nabla f^{\frac{p}{2}}\rangle \, dv + \underbrace{\int_{\mathbb{R}^3} f^{p+1} \, dv}_{\leq \varepsilon \int_{\mathbb{R}^3} \langle A[f]\nabla f^{\frac{p}{2}}, \nabla f^{\frac{p}{2}}\rangle \, dv + \frac{1}{\varepsilon}\int_{\mathbb{R}^3} f^p \, dv}$$

- Control on any $L^p$ norm for $f$ implies bound in $L^\infty$.
- $f$ in $L^\infty(\mathbb{R}^3 \times [0, T])$ implies $A[f]$ and $\nabla a[f]$ are bounded
- f is smooth up to time $T$ included.

□

IMPORTANT: The question whether $\|f\|_{L^\infty(0,T,L^{3/2^+}(\mathbb{R}^3))} \leq C$ is still open.

# Existing literature

We consider a regularization of the original Landau operator

$$\partial_t f = \mathrm{div} \int_{\mathbb{R}^3} \frac{\mathbb{P}(w-v)}{|w-v|^{-2-\gamma}} (f(w)\nabla_v f(v) - f(v)\nabla_w f(w))\, dw$$

with $-3 \leq \gamma \leq 0$.

- $\gamma \in [-2, 0)$: *moderately soft potential.* Existence of smooth solutions for general initial data.

# Existing literature

We consider a regularization of the original Landau operator

$$\partial_t f = \mathrm{div} \int_{\mathbb{R}^3} \frac{\mathbb{P}(w-v)}{|w-v|^{-2-\gamma}} (f(w)\nabla_v f(v) - f(v)\nabla_w f(w))\, dw$$

with $-3 \leq \gamma \leq 0$.

- $\gamma \in [-3, -2)$ *very soft potential.*
  - Existence of smooth solutions near equilibrium solution $e^{-|v|^2}$ (Guo 2000).
  - Existence of weak solutions (Villani, Desvillettes, ...).
  - Convergence towards Maxwellian as $t \to +\infty$ (Carrapatoso, Desvillettes, He '15)
  - Hausdorff dimension of the set of singular times is at most $\frac{1}{2}$ (Golse, G. Imbert, Vasseur '20)
  - No blow-up if $f \in L^{\frac{3^+}{5+\gamma}}(\mathbb{R}^3)$ uniformly in time (Silvestre '16, G. Guillen '19)
  - Non-existence of Type I self similar blow-up (Bedrossian, G., Snelson '21)
  - No blow-up of any kind after a finite time (Desvillettes, He, Jiang '21)

**IMPORTANT: global regularity is an open problem for any $\gamma < -2$.**

# A toy model: the Isotropic Landau equation

We consider the following **modification**:

$$\begin{aligned}\partial_t f &= \mathrm{div} \int_{\mathbb{R}^3} \frac{Tr\mathbb{P}(w-v)}{|w-v|}(f(w)\nabla_v f(v) - f(v)\nabla_w f(w))\, dw \\ &= \mathrm{div}(a[f]\nabla f - f\nabla a[f]) \\ &= a[f]\Delta f + f^2,\end{aligned}$$

where

$$a[f] := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(v-y)}{|y|}\, dy, \quad \Delta a[f] = -f.$$

We can interpret this equation as an isotropic version of the original Landau equation. This model was introduced by J. Krieger and R. Strain in 2012.

# The Isotropic Landau equation: $\gamma = -3$

## Theorem

*Consider the* isotropic *Landau equation with initial data $f_{in}$ such that*

$$\int_{\mathbb{R}^3} f_{in}(1, v, |v|^2, ln(f_{in}))\, dv < +\infty,$$

*and $f_{in}$ radially symmetric and monotonically decreasing. Then, any solution $f(v,t)$ satisfies*

$$\|f\|_{L^\infty(0,T,L^{3/2^+}(\mathbb{R}^3))} \leq C(T),$$

*and, consequently, $f \in C^\infty((0,T) \times \mathbb{R}^3)$ for all $T > 0$.*

# Global well-posedness for $\gamma = -3$: proof

## Proof.

We work with

$$M_f(r,t) := \int_{B_r(0)} f(v,t)\, dv,$$

which solves

$$\partial_t M_f = a[f]\partial_{rr} M_f + \frac{2}{r}\left(\frac{M_f}{8\pi r} - a[f]\right)\partial_r M_f.$$

The function $h(r) = r^2$ is a supersolution for $M_f(r)$ since,

$$a[f]\partial_{rr} h + \frac{2}{r}\left(\frac{M_f}{8\pi r} - a[f]\right)\partial_r h \leq 0.$$

Hence, $M_f(r) \leq r^2$ and

$$f(|v|,t) \leq \frac{M_f(|v|,t)}{|v|^3} \leq \frac{1}{|v|}.$$

□

# The Isotropic Landau equation with $-2.5 < \gamma < -2$

## Theorem

*Consider the* isotropic *Landau equation with* $-2.5 < \gamma < -2$*:*

$$\partial_t f = div(a[f]\nabla f - f\nabla a[f]) \qquad a[f] := c_\gamma \int_{\mathbb{R}^3} \frac{f(v-y)}{|y|^{-\gamma-2}}\, dy$$

$$= a[f]\Delta f + f\, h[f], \qquad h[f] := \tilde{c}_\gamma \int_{\mathbb{R}^3} \frac{f(v-y)}{|y|^{-\gamma}}\, dy$$

*The initial data* $f_{in}$ *is such that*

$$\int_{\mathbb{R}^3} f_{in}(1, v, |v|^2, ln(f_{in}))\, dv < +\infty,$$

*and* $f_{in} \in L^1 \cap L^p(\mathbb{R}^3)$ *for some* $p > \frac{3}{5+\gamma}$*. Then, any solution* $f(v,t)$ *satisfies*

$$\|f\|_{L^\infty(0,T,L^p(\mathbb{R}^3))} \leq C(T),$$

*and, consequently,* $f \in C^\infty((0,T) \times \mathbb{R}^3)$ *for all* $T > 0$*.*

# Hardy-type of inequality

Remeber

$$h[f] := \tilde{c}_\gamma \int_{\mathbb{R}^3} \frac{f(v-y)}{|y|^{-\gamma}}\, dy \quad \text{if } -3 < \gamma < -2, \qquad a[f] := c_\gamma \int_{\mathbb{R}^3} \frac{f(v-y)}{|y|^{-\gamma-2}}\, dy$$

**Lemma.** The following inequality holds

$$(3+\gamma) \int_{\mathbb{R}^3} h[f]\phi^2\, dv \le 4 \int_{\mathbb{R}^3} a[f] |\nabla \phi|^2\, dv, \quad \forall \phi \in C_0^1.$$

**Proof.** We start with classical Hardy's inequality. For $\gamma > -3$

$$(3+\gamma)^2 \int_{\mathbb{R}^3} \frac{\phi^2}{|v|^{-\gamma}}\, dv \le 4 \int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{|v|^{-2-\gamma}}\, dv$$

$$(3+\gamma)^2 \int_{\mathbb{R}^3} f(w) \int_{\mathbb{R}^3} \frac{\phi^2}{|v-w|^{-\gamma}}\, dvdw \le 4 \int_{\mathbb{R}^3} f(w) \int_{\mathbb{R}^3} \frac{|\nabla \phi|^2}{|v-w|^{-2-\gamma}}\, dvdw$$

# $L^p$ estimate: proof

## Proof.

Use the previous Hardy-type of inequality to obtain the good $L^p$-bound:

$$\frac{d}{dt}\int_{\mathbb{R}^3} f^p\,dv = -\frac{4(p-1)}{p}\int_{\mathbb{R}^3} a[f]|\nabla f^{p/2}|\,dv + \underbrace{(p-1)(-\gamma-2)\int_{\mathbb{R}^3} h[f]f^p\,dv}_{\leq \frac{4(p-1)(-\gamma-2)}{(3+\gamma)}\int_{\mathbb{R}^3} a[f]|\nabla f^{p/2}|\,dv}$$

$$\leq 0$$

if

$$1 \leq p \leq \frac{3+\gamma}{-2-\gamma}, \quad \text{assuming} \quad -\frac{5}{2} \leq \gamma < -2.$$

- Recall that for global-well-posedness it is sufficient to have $p > \frac{3}{5+\gamma}$.
- For $-\frac{5}{2}+\delta < \gamma < -2$, we have

$$\frac{3}{5+\gamma} < p \leq \frac{3+\gamma}{-2-\gamma}.$$

# Relevant literature for this talk:


- M.P. Gualdani and N. Guillen. *Hardy's inequality and the isotropic Landau equation.* J. Func. Anal. (2022).
- J. Bedrossian, M.P. Gualdani, S. Snelson. *Non-existence of some approximately self-similar singularities for the Landau, Vlasov-Poisson-Landau, and Boltzmann equations.* Trans. Amer. Math. Soc. (2022).
- F. Golse, M. P. Gualdani, C. Imbert, and A. Vasseur. *Partial regularity in time for the space homogeneous Landau equation with Coulomb potential.* Annales scientifiques de l' ENS (2022).
- M.P. Gualdani and N. Guillen. *On $A_p$ weights and the Landau equation.* Calc. Var. Partial Differential Equations 58 (2019).
- M.P. Gualdani and N. Guillen. *Estimates for radial solutions of the homogeneous Landau equation with Coulomb potential.* Analysis & PDE (2016).


**THANK YOU!**