MAT syllabus

The quadratic formula. Completing the square. Discriminant. Factorisation. Factor Theorem.

Revision

- The discriminant of a quadratic $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If the discriminant is positive then the quadratic has two real solutions. If the discriminant is zero then there's one (repeated) real solution. If the discriminant is negative then there are no real solutions.

- If $b^2 - 4ac \geqslant 0$, then the solution(s) of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- $ax^2 + bx + c$ can be written as $a(x - \alpha)(x - \beta)$ if $b^2 - 4ac \geqslant 0$, where α and β are roots given by the quadratic formula.

- (Complete the square) We can write $x^2 + bx + c$ in the form $(x + r)^2 + p$ because

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right).$$

  This is handy if we're trying to prove that the quadratic is non-negative, because anything squared is non-negative.

- (Difference of two squares) The expression $x^2 - a^2$ factorises as $(x - a)(x + a)$. This comes up quite a lot!

- (Factor Theorem) If $p(a) = 0$ for a polynomial $p(x)$, then $(x - a)$ is a factor of $p(x)$.

- The degree of a polynomial is the highest power of x, so the degree of any quadratic is 2, and the degree of any cubic is 3, for example.

- When sketching the graph of $y = ax^2 + bx + c$, we need to consider whether a is positive or negative (whether it's a "happy" or "sad" quadratic), whether the quadratic has any roots, and where it crosses the y-axis.

- Sometimes a function which is not a quadratic might secretly be a quadratic in a different variable. For example, $y = e^{2x} + e^{x+3} - 1$ is not a quadratic, but if we write $u = e^x$ then we have $y = u^2 + e^3 u - 1$, which is a quadratic. This is sometimes called "changing variable".

Warm-up

1. Find a positive number x which satisfies $x^2 = x + 1$.

2. Find a negative number x which satisfies $x^2 = x + 1$.

3. For which values of k does $x^2 - x + k = 0$ have exactly two real solutions?

4. For which values of k does $x^4 - x^2 + k = 0$ have exactly two real solutions?

5. Write $x^2 + 4x + 3$ in the form $(x + a)^2 + b$.

6. How many real solutions does $x^2 + bx + 1 = 0$ have? Find the different cases in terms of b.

7. Let $p(x) = x^3 - 13x^2 - 65x - 51$. Check that $p(17) = 0$. Factorise $p(x)$.

8. How many real solutions does the equation $3x^2 + 5x - 2 = 0$ have?

9. For what values of c does $3x^2 + 5x + c = 0$ have exactly two real solutions?

10. For what values of b does $3x^2 + bx - 2 = 0$ have exactly two real solutions?

11. For what values of a does $ax^2 + 5x - 2 = 0$ have exactly two real solutions?

12. What's the degree of the polynomial $(x^2 + 1)^{10}$? What's the degree of the polynomial $(x + 1)(x + 3)(x + 5)(x + 7) \ldots (x + 1727)(x + 1729)$?

13. Think about differentiating each of the polynomials in the previous question (but do not do it!) What would be the degree of the resulting polynomial be in each case?

14. In each of the following cases, choose a variable u in terms of x to make the function into a quadratic in u. There might be more than one sensible choice of u in each case.

- $y = x + \sqrt{x}$.
- $y = x^8 + 2x^4 + 1$.
- $y = \log_2(x^2) \times \log_2(2x^4)$.
- $y = e^{-2x} + 6e^{-4x}$.
- $y = \dfrac{1+x}{(1-x)^2}$.

Hint: This last one is really difficult, sorry!

MAT questions

MAT 2013 Q1A

For what values of the real number a does the quadratic equation

$$x^2 + ax + a = 1$$

have distinct real roots?

(a) $a \neq 2$, (b) $a > 2$, (c) $a = 2$, (d) all values of a.

Hint: Be careful here; a isn't in the normal place for a quadratic equation.

MAT 2016 Q1F

Let n be a positive integer. Then x^2+1 is a factor of

$$(3+x^4)^n - (x^2+3)^n(x^2-1)^n$$

for

(a) all n, (b) even n, (c) odd n, (d) $n \geqslant 3$, (e) no values of n.

Hint: change variable to make this into a more friendly polynomial.

MAT 2013 Q1E

The expression

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[(2x-1)^4(1-x)^5\right]+\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+1)^4(3x^2-2)^2\right]$$

is a polynomial of degree

(a) 9, (b) 8, (c) 7, (d) less than 7.

Hint: You do not need to be able to differentiate the terms inside the square brackets exactly, and you should not multiply out the brackets!

MAT 2014 Q2

Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0 \qquad (*)$$

(i) Show that if $x = 1$ is a solution of $(*)$ then

$$1 - \sqrt{2} \leqslant b \leqslant 1 + \sqrt{2}.$$

(ii) Show that there is no value of b for which $x = 1$ is a repeated root of $(*)$.

(iii) Given that $x = 1$ is a solution, find the value of b for which $(*)$ has a repeated root.

For this value of b, does the cubic

$$y = x^3 + 2bx^2 - a^2x - b^2$$

have a maximum or a minimum at its repeated root?

Hints: In the first part, once we've plugged in $x = 1$ we get an equation involving b and a^2. The variable a could be any real number, so all that I can say about a^2 is that it's $\geqslant 0$. That's the best I can do!

In the second part, once we've shown that $x = 1$ can't be a repeated root, then in the next case where $x = 1$ and the equation has a repeated root, that root cannot be $x = 1$- so the roots of the cubic must be 1 (not repeated) and something else (repeated). I suggest giving that "something else" a name (a letter).

For the last bit of part (iii), a sketch is probably helpful. At this stage, we know where all the roots are, and we know the general shape of the cubic.

Extension

The following material is included for your interest only, and not for MAT preparation.

Let's try to factorise $p(x) = x^4 - a^4$. We can see that a is a root, so by the factor theorem, we can write $p(x) = (x-a)q(x)$ for some cubic $q(x)$. Long division gives $q(x) = x^3+ax^2+a^2x+a^3$.

Now $-a$ is a root of that cubic, so $q(x) = (x + a)r(x)$ where $r(x)$ is some quadratic. Long division gives $r(x) = x^2 + a^2$. The discriminant of this quadratic is $-4a^2$, which is negative, so there are no more linear factors. Alternatively, note that $x^4 - a^4 = (x^2)^2 - (a^2)^2$ is the difference of two squares, so $p(x) = (x^2 - a^2)(x^2 + a^2)$. The first term here is also the difference of two squares, so $p(x) = (x - a)(x + a)(x^2 + a^2)$. The difference of two squares is so useful! Now that you've seen this, try factorising $x^8 - a^8$ and $x^{16} - a^{16}$ (and so on?).

As well as the Factor Theorem, there's also something called the Remainder Theorem. The Remainder Theorem states that when the polynomial $p(x)$ is divided by $(x - a)$ then the remainder is $p(a)$. The Factor Theorem is then a special case of the Remainder Theorem (when a is a root).

Here are a couple of previous MAT questions based on the Remainder Theorem. Just to be clear, the MAT syllabus was changed in 2018 and the Remainder Theorem was removed, so these would not be suitable MAT questions after 2017.

MAT 2013 Q1G

Let $n \geqslant 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x - 1) + (x - 2) + \ldots (x - n).$$

What is the remainder when $p_n(x)$ is divided by $p_{n-1}(x)$?

(a) $\frac{n}{2}$, (b) $\frac{n+1}{2}$, (c) $\frac{n^2+n}{2}$, (d) $\frac{-n}{2}$.

MAT 2017 Q1H

In this question a and b are real numbers, and a is non-zero.
When the polynomial $x^2 - 2ax + a^4$ is divided by $x + b$ the remainder is 1.
The polynomial $bx^2 + x + 1$ has $ax - 1$ as a factor.

It follows that b equals

(a) 1 only, (b) 0 or -2, (c) 1 or 2, (d) 1 or 3, (e) -1 or 2.