MAT syllabus

Sequences defined iteratively and by formulae. Arithmetic and geometric progressions*. Their sums*. Convergence condition for infinite geometric progressions*.
* Part of full A-level Mathematics syllabus.

Revision

- A sequence a_n might be defined by a formula for the n^{th} term like $a_n = n^2 - n$.
- A sequence a_n might be defined with an relation like $a_{n+1} = f(a_n)$ for $n \geqslant 0$, if we're given the function $f(x)$ and also given a first term like $a_0 = 1$. (The "first term" might be a_0 if we feel like counting from zero).
- The sum of the first n terms of a sequence a_k can be written with the notation $\displaystyle\sum_{k=0}^{n-1} a_k$ (if the first term is a_0) or $\displaystyle\sum_{k=1}^{n} a_k$ (if the first term is a_1).
- An arithmetic sequence is one where the difference between terms is constant. The terms can be written as a, $a+d$, $a+2d$, $a+3d$, $\dots$, where a is the first term and d is the common difference.
- The sum of the first n terms of an arithmetic sequence with first term a and common difference d is $\dfrac{n}{2}\left(2a + (n-1)d\right)$, which you can remember as "first term plus last term, times the number of terms, divided by two".
- A geometric sequence is one where the ratio between consecutive terms is constant. The terms can be written as a, ar, ar^2, ar^3, $\dots$ where a is the first term and r is the common ratio.
- The sum of the first n terms of a geometric sequence with first term a and common ratio r is $\dfrac{a(1-r^n)}{1-r}$. One way to remember this is to remember what happens if we multiply the sum of the first n terms of a geometric series by $(1-r)$,
$$\begin{aligned}(1-r)(a + ar + \cdots + ar^{n-1}) =& (a - ar) + (ar - ar^2) + \cdots + (ar^{n-1} - ar^n)\\ =& a - ar^n.\end{aligned}$$
- For a geometric sequence a_n, the sum to infinity is written as $\displaystyle\sum_{k=0}^{\infty} a_k$. If the common ratio r satisfies $|r| < 1$ then this is equal to $\dfrac{a}{1-r}$. If $|r| \geqslant 1$ then this sum to infinity does not converge (it does not approach any particular real number).

Revision Questions

1. A sequence is defined by $a_n = n^2 - n$. What is a_3? What is a_{10}? Find $a_{n+1} - a_n$ in terms of n. Find $a_{n+1} - 2a_n + a_{n-1}$ in terms of n.

2. A sequence is defined by $a_0 = 1$ and $a_n = a_{n-1} + 3$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}.

3. A sequence is defined by $a_0 = 1$ and $a_n = \frac{a_{n-1}}{3}$ for $n \geqslant 1$. Find $a_0 + a_1 + \cdots + a_{10}$. Find a_{1000}. Does the sum of all the terms of this sequence converge? If it does, what is the sum to infinity?

4. A sequence is defined by $a_0 = 1$ and $a_n = 3a_{n-1} + 1$ for $n \geqslant 1$. A sequence b_n is defined by $b_n = A \times 3^n + B$ where A and B are real numbers. Find values for A and B such that $a_n = b_n$ for all $n \geqslant 0$.

5. A sequence is defined by $a_n = An^2 + Bn + C$ where A, B, and C are real numbers. Find A, B, and C in terms of a_0, a_1, and a_2.

6. When does the sum $1 + x^3 + x^6 + x^9 + x^{12} + ...$ converge? Simplify it in the case that it converges.

7. When does the sum $2 - x + \dfrac{x^2}{2} - \dfrac{x^3}{4} + \ldots$ converge? Simplify it in the case that it converges.

8. If the first term of an arithmetic progression is 5 and the common difference is 3, what is the 15^{th} term?

9. The sum of the first k terms of an arithmetic progression is equal to the sum of the next k terms. What can you deduce?

10. If the sum of the first n terms of an arithmetic progression is $3n^2 + 5n$, what is the n^{th} term?

11. What is the sum of the first 100 positive even integers (starting at 2)?

12. The first term of a geometric progression is 3 and the third term is 27. Find two possibilities for the sum of the first 5 terms.

13. A sequence is defined by $a_0 = 3$ and then for $n \geqslant 1$ a_n is the sum of all previous terms. Find a_n in terms of n for $n \geqslant 1$.

14. A sequence is defined by $C_0 = 1$ and then for $n \geqslant 0$,

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

Find C_1 and C_2 and C_3 and C_4.

MAT questions

MAT 2008 Q2

(i) Find a pair of positive integers, x_1 and y_1, that solve the equation

$$(x_1)^2 - 2(y_1)^2 = 1.$$

(ii) Given integers a, b, we define two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ by setting

$$x_{n+1} = 3x_n + 4y_n, \qquad y_{n+1} = ax_n + by_n, \qquad \text{for } n \geqslant 1.$$

Find *positive* values for a, b such that

$$(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2 .$$

(iii) Find a pair of integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$.

(iv) Using the values of a and b found in part (ii), what is the approximate value of x_n/y_n as n increases?

[See the next page for hints]

Hints

(i) Searching small values of x_1 or small values of y_1 is a good idea here. We're only asked to find a pair, not all such pairs. The question doesn't specific whether zero counts as a positive number (some people do count it, some people don't), so that's up to you.

(ii) Substitute everything in and hope for the best. We want this to be true for lots of different values of x_n and y_n (presumably), so we might aim to do something like comparing coefficients.

Hopefully this will give us some equations involving a and b. We're not too worried about finding all possible solutions here; we're just looking for anything that works, and that has a and b positive.

(iii) This part of the question is all about understanding the previous part. We found a way to generate a sequence x_n and a sequence y_n, and we showed that the sequences satisfy some sort of rule. Why did we do that? What's it got to do with the value of $X^2 - 2Y^2$?

It's easy to get distracted by the relationship that we've just proved if you're looking for a link between x_{n+1} and x_n. Don't forget that we also have rules like $x_{n+1} = 3x_n + 4y_n$ which are easier to work with if we want to calculate x_{n+1} from our knowledge of previous values of x_n and y_n.

Alternatively, try large numbers Y until you find one with $2Y^2 + 1$ equal to a square number. This might take a while!

(iv) From our work on the previous parts, we know that x_n and y_n satisfy a particular equation. We also know that x_n and y_n will be large for large n. Can you see how to convert the equation you've got into a fact about x_n/y_n?

Extension

[Just for fun, not part of the MAT question]

- Find some rational approximations to $\sqrt{3}$ with a similar method.

MAT 2012 Q5
A particular robot has three commands;
F: Move forward a unit distance;
L: Turn left 90°
R: Turn right 90°

A *program* is a sequence of commands. We consider particular programs P_n (for $n \geqslant 0$) in this question. The basic program P_0 just instructs the robot to move forward:

$$P_0 = \mathbf{F}.$$

The program P_{n+1} (for $n \geqslant 0$) involves performing P_n, turning left, performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n\mathbf{L}P_n\mathbf{R}.$$

So, for example, $P_1 = \mathbf{FLFR}$.

(i) Write down the program P_2.

(ii) How far does the robot travel during the program P_n? In other words, how many **F** commands does it perform?

(iii) Let l_n be the total number of commands in P_n; so, for example, $l_0 = 1$ and $l_1 = 4$.

Write down an equation relating l_{n+1} to l_n. Hence write down a formula for l_n in terms of n. **Hint:** consider $l_n + 2$.

(iv) The robot starts at the origin, facing along the positive x-axis. What direction is the robot facing after performing the program P_n?

(v) Draw the path the robot takes when it performs the program P_4.

(vi) Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$. Give an equation relating (x_{n+1}, y_{n+1}) to (x_n, y_n)

What is (x_8, y_8)? What is (x_{8k}, y_{8k})?

[See the next page for hints]

Hints

(i) Extend what we've learned about P_1 to P_2. I suppose $P_2 = P_1\mathbf{L}P_1\mathbf{R}$ but we can do better than that!

(ii) How many **F** commands are there in P_0, P_1, and P_2?

(iii) l_{n+1} is the length of P_{n+1}. How long are P_0, P_1, and P_2? From the definition $P_{n+1} = P_n\mathbf{L}P_n\mathbf{R}$, what would you expect the length of P_3?

It's hard to solve this recursion relation using MAT-level maths. Luckily, there's a hint. We could give the sequence $l_n + 2$ a name like a_n and try to convert our fact about l_{n+1} and l_n into a fact about a_{n+1} and a_n.

(iv) How many **L** commands are there in P_n? How many **R** commands are there in P_n?

(v) Draw paths for P_1 and P_2 and P_3 first.

Remember that the robot spins on the spot for **L** and for **R**.

Remember that, during P_2, the robot turns at the end of P_1 and then immediately turns again for the **L** before the next P_1 starts.

At the end of each P_n, the robot is facing in a particular direction which might or might not be the direction of the most recent **F** command that it moved (it might have done some spinning at the end).

(vi) Here are two things to think about.

- What would happen if we started the robot at (a, b) and ran program P_n?
- What would happen if we started the robot at $(0, 0)$ and ran $\mathbf{L}P_n$?

Find (x_2, y_2) and (x_3, y_3) and so on up to (x_8, y_8). If you spot any shortcuts, take them!

Describe a simpler program **Q** that takes the robot from $(0, 0)$ to (x_8, y_8) (literally a shortcut for the robot to take). Explain why (x_9, y_9) is the same position that a robot would end up in if it ran the program **QLQR**. Why is this a bit like the calculations you just did for (x_2, y_2) and (x_3, y_3)?