From the N-Body Schrödinger Equation to the Euler-Poisson System

François Golse

École polytechnique, CMLS

Oxbridge PDE Conference 2022
Mathematical Institute, University of Oxford
April 11th-13th 2022

Work in collaboration with T. Paul
Comm. on Pure and Appl. Math., to appear (2022)

The Problem

At present, no rigorous derivation of the Vlasov-Poisson system in space dimension three from Newton's motion equations for a gas of charged particles with (repulsive) Coulomb interaction.

Recent progress by S. Serfaty and M. Duerinckx (DMJ2020) in the special case of monokinetic distribution functions

In the quantum setting, the Hartree equation with Coulomb interaction has been derived from the N-body Schrödinger equation by a mean-field limit (L. Erdös-H.T. Yau 2003, subsequent contributions by Rodnianski-Schlein CMP2009, Pickl LMP2009, Knowles-Pickl, Benedikter, Porta, Schlein, Saffirio....)

Problem what about the **joint** mean-field (particle number $N \gg 1$) and classical ($\hbar \ll 1$) limits?

Consider the N-body **Schrödinger equation**

$$i\hbar\partial_t\Psi_{\hbar,N}(t,X_N)=\mathcal{H}_N\psi_N(t,X_N)\,,\qquad \Psi_{\hbar,N}\Big|_{t=0}=(\psi_\hbar^{in})^{\otimes N}$$

with the notation

$$X_N:=(x_1,\ldots,x_N)\in(\mathbf{R}^3)^N$$

By Kato's Thm (TAMS1951) **quantum Hamiltonian** on $L^2(\mathbf{R}^{3N})$

$$\mathcal{H}_N=\sum_{k=1}^N-\tfrac{1}{2}\hbar^2\Delta_{x_k}+\tfrac{1}{N}\sum_{1\le k<l\le N}\tfrac{1}{|x_k-x_l|}=\mathcal{H}_N^*$$

By Stone's theorem, the N-**body wave function** is

$$\Psi_{\hbar,N}(t,\cdot)=e^{-it\mathcal{H}_N/\hbar}(\psi_\hbar^{in})^{\otimes N}\quad\text{with }(\psi_\hbar^{in})^{\otimes N}(X_N)=\prod_{k=1}^N\psi_\hbar^{in}(x_k)$$

The Pressureless Euler-Poisson System

Unknown $\rho(t,x) \geq 0$ (density) and $u(t,x) \in \mathbf{R}^3$ (velocity field)

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho u) = 0\,, & \rho\big|_{t=0} = \rho^{in} \\ \partial_t u + u \cdot \nabla_x u = -\nabla_x \frac{1}{|x|} \star_x \rho\,, & u\big|_{t=0} = u^{in} \end{cases}$$

If (ρ, u) is a **classical solution** of the pressureless Euler-Poisson system, the monokinetic distribution function

$$f(t,x,\xi) := \rho(t,x)\delta(\xi - u(t,x))$$

is a solution of the **Vlasov-Poisson system**

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f - \nabla_x V_f(t,x) \cdot \nabla_\xi f = 0 \\ -\Delta_x V_f(t,x) = 4\pi \int_{\mathbf{R}^3} f(t,x,\xi) d\xi \end{cases}$$

Local Existence/Uniqueness Theorem for Euler-Poisson

Let $u^{in} \in L^\infty(\mathbf{R}^3)$ be s.t. $\nabla_x u^{in} \in H^{2m}(\mathbf{R}^3)$, and $\rho^{in} \in H^{2m}(\mathbf{R}^3)$ s.t.

$$\rho^{in}(x) \geq 0 \text{ for a.e. } x \in \mathbf{R}^3, \quad \text{and} \quad \int_{\mathbf{R}^3} \rho^{in}(y)dy = 1$$

(1) There exists $T \equiv T[\|\rho^{in}\|_{H^{2m}(\mathbf{R}^3)} + \|\nabla_x u^{in}\|_{H^{2m}(\mathbf{R}^3)}] > 0$, and a unique solution (ρ, u) of the Euler-Poisson system s.t.

$$u \in L^\infty([0,T] \times \mathbf{R}^3) \quad \text{while } \rho \text{ and } \nabla_x u \in C([0,T], H^{2m}(\mathbf{R}^3))$$

(2) Besides, for all $t \in [0,T]$, one has

$$\rho(t,x) \geq 0 \text{ for a.e. } x \in \mathbf{R}^3, \quad \text{and} \quad \int_{\mathbf{R}^3} \rho(t,y)dy = 1$$

Thm [F.G.-T. Paul CPAM2022]
Let $\rho^{in} \in H^4(\mathbf{R}^3) \cap \mathcal{P}(\mathbf{R}^3)$ and $u^{in} \in L^\infty(\mathbf{R}^3)^3$ s.t. $\nabla u^{in} \in H^4(\mathbf{R}^3)^9$.
Let (ρ, u) be the (classical) solution on $[0, T] \times \mathbf{R}^3$ for some $T > 0$ of the pressureless Euler-Poisson system initial data (ρ^{in}, u^{in}).

Let $\Psi^{in}_{\hbar,N} = (\psi^{in}_\hbar)^{\otimes N}$, with $\|\psi_\hbar\|_{L^2} = 1$ satisfying

$$\sup_{0<\hbar<1} \|\hbar^2 \Delta_x \psi^{in}_\hbar\|_{L^2(\mathbf{R}^3)} < \infty \,, \qquad \lim_{\hbar\to 0^+} \||i\hbar\nabla_x + u^{in}|^2 \psi^{in}_\hbar\|_{L^2(\mathbf{R}^3)} = 0$$

$$\lim_{\hbar\to 0} \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{(|\psi^{in}_\hbar(x)|^2 - \rho^{in}(x))(|\psi^{in}_\hbar(y)|^2 - \rho^{in}(y))}{|x-y|} dxdy = 0$$

Remark test these assumptions on WKB initial states of the form

$$\psi^{in}(x) = a^{in}(x) \exp\left(\frac{iS^{in}(x)}{\hbar}\right) \,, \qquad |a^{in}|^2 = \rho^{in} \,, \quad u^{in} = \nabla S^{in}$$

Let $\mathcal{H}_N :=$ N-body Hamiltonian with Coulomb potential and set

$$\Psi_{\hbar,N}(t) := e^{-\frac{it\mathcal{H}_N}{\hbar}}\Psi^{in}_{\hbar,N}$$

Then, in the limit as $\hbar + \frac{1}{N} \to 0$, one has

$$\int_{\mathbf{R}^{3(N-1)}} |\Psi_{\hbar,N}(t,\cdot,X_{2,N})|^2 dX_{2,N} \to \rho(t,\cdot)$$
$$\hbar \int_{\mathbf{R}^{3(N-1)}} \operatorname{Im}\left(\overline{\Psi_{\hbar,N}}\nabla_{x_1}\Psi_{\hbar,N}\right)(t,\cdot,X_{2,N})dX_{2,N} \to \rho u(t,\cdot)$$

for the narrow topology of Radon measures on $\mathbf{R}^3$, with the notation

$$X_{2,N} = (x_2,\ldots,x_N)$$

For all $\rho \in L^\infty(\mathbf{R}^3)$, all $u \in W^{1,\infty}(\mathbf{R}^3)^3$ and all $X_N \in \mathbf{R}^{3N}$, set

$$\begin{cases} F[X_N,\rho] := \iint_{x\neq y} \frac{(\mu_{X_N}-\rho)(dx)(\mu_{X_N}-\rho)(dy)}{|x-y|} \\ G[X_N,\rho,u] := \iint_{x\neq y} \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|^3}(\mu_{X_N}-\rho)(dx)(\mu_{X_N}-\rho)(dy) \end{cases}$$

There exists $C>2$ such that, for all $\rho \in L^\infty(\mathbf{R}^3)$, all $u \in W^{1,\infty}(\mathbf{R}^3)^3$ and a.e. $X_N \in \mathbf{R}^{3N}$

$$|G[X_N,\rho,u]| \leq C\|\nabla u\|_{L^\infty} F_N[X_N,\rho] + \frac{C}{N^{1/3}}(1+\|\rho\|_{L^\infty})(1+\|u\|_{W^{1,\infty}})$$

Besides, there exists $C'>0$ such that

$$F[X_N,\rho] \geq -\frac{C'}{N^{2/3}}(1+\|\rho\|_{L^\infty(\mathbf{R}^3)})$$

With the notation $J_1 A = A \otimes \overbrace{I \otimes \ldots \otimes I}^{N-1 \text{ terms}}$, we consider the quantity

$$\mathcal{E}[\Psi_{\hbar,N}, \rho, u](t) := \left\langle \Psi_{\hbar,N}(t) \right| J_1 |-i\hbar\nabla_x - u(t,\cdot)|^2 \left| \Psi_{\hbar,N}(t) \right\rangle$$

$$+ \left\langle \Psi_{\hbar,N}(t) \right| F[X_N, \rho(t,\cdot)] \left| \Psi_{\hbar,N}(t) \right\rangle$$

Denoting $\Sigma := \frac{1}{2}(\nabla_x u + (\nabla_x u)^T)$ the deformation tensor, one has

$$\frac{d}{dt}\mathcal{E}[\Psi_{\hbar,N}, \rho, u](t) + 2 \left\langle \Psi_{\hbar,N} \right| J_1((i\hbar\nabla_x + u)^T \Sigma (i\hbar\nabla_x + u)) \left| \Psi_{\hbar,N} \right\rangle$$

$$= \tfrac{1}{2}\hbar^2 \left\langle \Psi_{\hbar,N} \right| J_1(\Delta_x \operatorname{div}_x u(t,\cdot)) \left| \Psi_{\hbar,N} \right\rangle + \left\langle \Psi_{\hbar,N} \right| G[X_N, \rho, u] \left| \Psi_{\hbar,N} \right\rangle$$

Notation if $\psi \in L^2(\mathbf{R}^d)$ and A is an operator on $L^2(\mathbf{R}^d)$, we denote

$$\langle \psi | A | \psi \rangle := \int_{\mathbf{R}^d} \overline{\psi(x)} (A\psi)(x) dx$$

By Gronwall's and Serfaty's inequalities, one arrives at the bound

$$
\begin{aligned}
0 \leq \mathcal{E}[\Psi_{\hbar,N},\rho,u](t) + \frac{C'}{N^{2/3}}(1+\|\rho\|_{L^\infty(\mathbf{R}^3)})& \\
\leq e^{CT\|\nabla u\|_{L^\infty}}\left(\underbrace{\mathcal{E}[\Psi_{\hbar,N},\rho,u](0)}_{\to 0}+\frac{C'}{N^{2/3}}(1+\|\rho\|_{L^\infty(\mathbf{R}^3)})\right)& \\
+Te^{CT\|\nabla u\|_{L^\infty}}\frac{C}{N^{1/3}}(1+\|\rho\|_{L^\infty})(1+\|u\|_{W^{1,\infty}})& \\
+Te^{CT\|\nabla u\|_{L^\infty}}\tfrac{1}{2}\hbar^2\|\Delta_x \operatorname{div}_x u\|_{L^\infty}&
\end{aligned}
$$

By the lower bound in Serfaty's inequality and the Cauchy-Schwarz inequality

$$
\begin{aligned}
\mathcal{E}[\Psi_{\hbar,N},\rho,u](t) + \frac{C'}{N^{2/3}}(1+\|\rho\|_{L^\infty(\mathbf{R}^3)})& \\
\geq \left|\left\langle \Psi_{\hbar,N}(t)\right| J_1(-i\hbar\nabla_x - u(t,\cdot))\left|\Psi_{\hbar,N}(t)\right\rangle\right|^2&
\end{aligned}
$$

Starting from the decomposition (good exercise...)

$$\frac{1}{4\pi|x-y|}=\int_0^\infty dr\int_{\mathbf{R}^3} G_r(x-z)G_r(y-z)dz\,,\qquad G_r(w):=\frac{e^{-\frac{|w|^2}{2r}}}{(2\pi r)^{\frac{3}{2}}}$$

one can prove that

$$\begin{aligned}&\int_\epsilon^\infty \|e^{r\Delta/2}\rho_{\hbar,N:1}(t,\cdot)-\rho(t,\cdot)\|_{L^2}^2 dr\\ \le\ &\underbrace{\langle\Psi_{\hbar,N}(t)|\,F[X_N,\rho(t,\cdot)]\,|\Psi_{\hbar,N}(t)\rangle}_{\to 0}+O(N^{-2/3})\end{aligned}$$

where

$$\rho_{\hbar,N:1}(t,\cdot):=\int_{\mathbf{R}^{3(N-1)}}|\Psi_{\hbar,N}(t,\cdot,X_{2,N})|^2 dX_{2,N}$$

Conclusions/Perspectives

(1) Using the quantum analogue of phase-space empirical measures in [F.G. - T. Paul: Commun. Math. Phys. 369 (2019), 1021-1053] leads to a proof which parallels the Serfaty-Duericnkx derivation of the Euler-Poisson system [Duke Math. J. 169 (2020), 2887-2935] for a classical Coulomb gas. The proof above avoids this technical machinery by a duality approach.

(2) The classical limit of the Schrödinger-Poisson to the Vlasov-Poisson system can be proved by the same method. This extends Thm IV.5 in [Lions-Paul, Rev. Mat. Iberoam. 1993] to the case of pure states (with singular — Dirac type — Wigner measure).

(3) Extensions to non monokinetic situations (in the regime where both $\hbar$ and $1/N$ are small independently) remain a challenging open problem