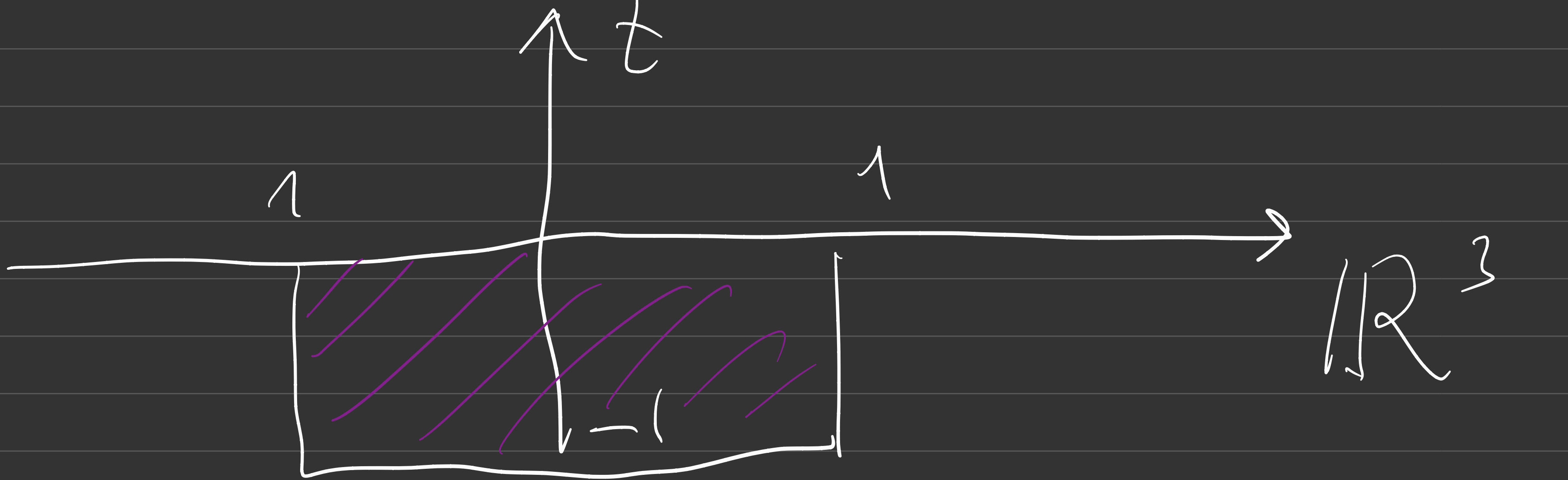
About work of my PhD students: Tobias Barker and Francis Hounkpe

Question

Regularity of weak solutions to the 3-dimensional Navier-Stokes System

$$\begin{cases} \partial_t v + v \cdot \nabla v - \nu \Delta v = -\nabla q \\ \operatorname{div} v = 0 \end{cases}$$

in $Q = B \times]-1, 0[$



Scaling

$$v^\lambda(y,s) = \lambda v(x,t)$$

$$q^\lambda(y,s) = \lambda^2 q(x,t)$$

$$x = x_0 + \lambda y, \quad t = t_0 + \lambda^2 s$$

$$\lambda > 0$$

Starting properties

(i) $\gamma = 1$

(ii) $v \in L_{2,\infty}(Q) = L_\infty(-1,0; L_2(B))$

$\nabla v \in L_2(Q)$

$q \in L_{3/2}(Q)$

motivated by properties of weak Leray-Hopf solutions to initial boundary value problems for Navier-Stokes equations.

Main estimate

(Energy)

$$\int_B \varphi(x,t)\,|v(x,t)|^2 + 2\int_{-1}^{t}\int_B \varphi\,|\nabla v|^2\,dx\,dt$$

$$\leq \int_Q |v|^2\left(\partial_t\varphi + \Delta\varphi\right) + v\cdot\nabla\varphi\left(|v|^2 + 2q\right)$$

$$\varphi > 0$$

Supercriticality

$$\int_{B\left(\frac{a}{\lambda}\right)} |v^\lambda(y,s)|^2\,dy + \int_{-\left(\frac{a}{\lambda}\right)^2}^{s} \int_{B\left(\frac{a}{\lambda}\right)} |\nabla v^\lambda|^2\,dy\,ds'$$

$$= \frac{1}{\lambda}\int_{B(a)} |v(x,t)|^2\,dx + \int_{-a^2}^{t}\int_{B(a)} |\nabla v|^2\,dx\,dt' \xrightarrow[\lambda\to 0]{} \infty$$

$\lambda y = x,\quad \lambda^2 s = t$

Critical Spaces

in $\mathbb{R}^3$

$$\|v^\lambda\|_{\mathcal{B}} = \|v\|_{\mathcal{B}}$$

$$\dot{H}^{1/2} \subset L_3 \subset L^{3,\infty} \subset BMO^{-1}$$

$$\subset B^{-1}_{\infty,\infty}$$

Tobias Barker

Consider Cauchy Problem

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q$$

$$\operatorname{div} v = 0$$

in $Q_+ = \mathbb{R}^3 \times]0, \infty[$

$$v(x,t) \xrightarrow[|x| \to \infty]{} 0$$

$$v(\cdot, 0) = v_0(\cdot) \subset B$$

Typical results on local well-posedness say that

$\exists\, T_0 = T(\|v_0\|_B) > 0$ such that

v solution is unique on $]0, T_0[$.

Spaces B, for example,

H^1, L_p with $p > 3$.

This solution, of course, coincides with weak Leray – Hopf solution if such a solution exists for given initial data.

It is an interesting question whether such time $T_0 = T(\|v_0\|_B)$ exists if B is a critical space. If so, we could argue as follows. Take arbitrary $T_* > 0$, find $\lambda = T_0(\|v_0\|_B)/T_*$, consider $v^\lambda(y,s) = \lambda v(x,t)$.

$x = \lambda y$
$t = \lambda^2 s$

We know that solution v^λ exists and unique on $]0, T_0(\|v_0^\lambda\|_B)[$

But $v(x,t)$ exists and unique then on $]0, \lambda T_0(\|v_0^\lambda\|_B)[=$

$$=]0, \lambda T_0(\|v_0\|_B)[=]0, T_*[$$

So, for critical case, T_0 should depend on not only $\|v_0\|_B$ but something else like modulus of continuity, etc.

Related question which is critical norm is blowing up which is not.

We know $\|v(\cdot,t)\|_{\dot{H}^{1/2}}$, $\|v(\cdot,t)\|_{L_3}$ must blow up if there are singular points.

As to other norms like $L^{3,\infty}$, BMO^{-1}, etc, it is an open question

Tobias considers the case of $L^{3,\infty}$ based on earlier results of G.S and V. Sverak

The idea is simple:

Let
$$v^1 - \Delta v^1 + \nabla q^1 = 0$$
$$\operatorname{div} v^1 = 0$$
$$v^1\big|_{t=0} = v_0 \in L^{3,\infty}$$

$\Rightarrow$ $v^2 = v - v^1$ is a perturbation with $v^2\big|_{t=0} = 0$

The idea is that v^2 satisfies all the properties like weak Leray-Hopf solution, i.e.

1) exists for $t > 0$
2) satisfies global energy inequality
3) satisfies local energy inequality $\Longrightarrow$ CKN theory applicable

Such solution is called weak $L^{3,\infty}$ - solution

Pro

1) Global existence

2) weak stability

$$\begin{tikzcd} v_0^m \arrow[r, rightharpoonup, "*"] \arrow[d, bend right] & v_0 \arrow[d, bend left] \\ v^m \arrow[r] & v \end{tikzcd}$$

in $L^{3,\infty}$

weak $L^{3,\infty}$-solution

3) stronger then local energy solution given

G.-P. Lemarié-Rieusset

Contra

Also existence is global

$L^{3,\infty}$ might blow up.

Difficult part: energy estimates

for Γ^2

Francis Hounkpe

Supercritical Nature of the Navier-Stokes equations

Toy - model

Otelbaev (Wrong Proof)

Error has been discovered

N. Filonov on page 55

Terence Tao gave counter-example showing that energy estimate plus skew symmetry are not enough for well-posedness.

But the NSE's has additional special structure that leads to vorticity equations

$$\begin{cases} \partial_t w + u \cdot \nabla w - \Delta w = w \cdot \nabla u \\ w = \operatorname{rot} u \end{cases}$$

In Terence Tao's counter-example, there is no such structure.

H. Jia, V. Sverak proved some conditional result saying that one can construct non-uniqueness (instantaneous) with help of forward self-similar solutions.

It is interesting to look at another super-critical models to understand the role of supercriticality and the pressure.

Toy models

$$\partial_t v + v\cdot\nabla v + \frac{v}{2}\,\mathrm{div}\, v - \varkappa \nabla \mathrm{div}\, v - \Delta v = 0$$

Marginal cases

$\varkappa = 0$ and $\varkappa \longrightarrow \infty$

In the last cases, we have an approximation of classical NS Eqs. Moreover, Francis showed that similar things and even better take place for forward self-similar solutions.

For NSE's,

$$v(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right),$$

where profile U satisfies

(v_0 must be -1-homogeneous, $v(x) = \lambda v(\lambda x)$, $\lambda > 0$)

$$\begin{cases} -\Delta U + U\cdot\nabla U - \frac{1}{2}U - x\cdot\nabla U = -\nabla P \\ \operatorname{div} U = 0 \\ |U(x) - v_0(x)| = o\left(\frac{1}{|x|}\right) \qquad |x| \to \infty \end{cases}$$

For the toy model, we have

$$\begin{cases} -\Delta U^\varkappa - \varkappa \operatorname{div} U^\varkappa + U^\varkappa\cdot\nabla U^\varkappa + \\ \quad + \frac{1}{2}U^\varkappa \operatorname{div} U^\varkappa - \frac{1}{2}U^\varkappa - x\cdot\nabla U^\varkappa = 0 \\ |U^\varkappa(x) - v_0(x)| = o\left(\frac{1}{|x|}\right) \\ |x| \to \infty \end{cases}$$

It turns out to be that $v^{\varepsilon}(x,t) = \frac{1}{\sqrt{t}}\, U^{\varepsilon}\left(\frac{x}{\sqrt{t}}\right)$ is weak $L^{3,\infty}_{1}$ - solution to the toy model

$$\partial_t v^{\varepsilon} - \Delta v^{\varepsilon} + v^{\varepsilon}\cdot\nabla v^{\varepsilon} + \frac{1}{2} v^{\varepsilon} \operatorname{div} v^{\varepsilon} - \varepsilon\nabla \operatorname{div} v^{\varepsilon} = 0$$

$$v^{\varepsilon}\big|_{t=0} = v_0$$

$$v^\alpha = u^\alpha + w^\alpha \longrightarrow V^\alpha = U^\alpha + W^\alpha$$

$$\partial_t u^\alpha - \Delta u^\alpha - \alpha \nabla \operatorname{div} u^\alpha = 0$$

$$\|$$

$$u^\alpha$$

In the same way

$$v = u + w \longrightarrow V = U + W$$

$$\partial_t u - \Delta u = 0$$

$$u|_{t=0} = v_0$$

Francis showed at least that $W^\alpha \longrightarrow W$ in certain sense

(The same of course is true U^k and U even higher derivatives).