Zero viscosity limit for solutions of the Navier-Stokes Equations in a domain with curved boundary and no slip boundary condition
Report on a joint work with To. Nguyen, Tr. Nguyen and E. Titi.
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The convergence of the solution of u_ν of the Navier -Stokes equations, with no slip boundary condition, to the solution of the Euler equations generates a boundary layer because the tangential component of the velocity does not remain equal to 0 .

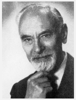
In $\Omega \times \mathbb{R}_t^+ \quad \partial_t u_\nu + u_\nu \cdot \nabla u_\nu + \nabla p_\nu = \nu \Delta u_n u \quad \partial_t u + u \cdot \nabla u + \nabla p = 0\,,$

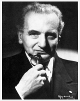
$\nabla \cdot u_\nu = \nabla \cdot u = 0\,.$

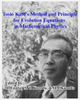
On $\partial\Omega \quad u_\nu = 0$ and $u \cdot \vec{n} = 0\,.$

First observed by Prandlt who proposed in 1904 the eponym equations, based on a parabolic scaling with a boundary layer of the order of $\sqrt{\nu}$*. They became the paradigm of boundary layers analysis in many even less subtle situations (fully linear problems!!) . However the non linearity may generate "turbulence"* **propagating in the bulk of the fluid** *such that the limit may seriously differ from the solution of the Euler equation . Von Karman and Prandlt suggested in 1920 that the origin of such singular behavior is in the eponym turbulent boundary layer of the order of* ν *much smaller than* $\sqrt{\nu}$*.*

A milestone in our community is the paper of Kato 1984. Kato does not prove or disprove the convergence of the solution of the Navier-Stokes equations to the solution of the Euler equations, but he shows how this issue is related to several "physical " issues like the absence of anomalous energy dissipation or the generation of vorticity in a boundary layer of order ν. In 213 E. Titi and C.B. proposed an avatar of the theorem of Kato, based on simple Gronwall estimate . ie Theorem 1







Theorem

(In dimension 2 and 3) Let u be weak solution to the Euler equations in $[0, T] \times \Omega$ *satisfying* $\|\nabla u\|_{L^\infty([0,T]\times\Omega)} < \infty$. *Consider* $(\nu > 0\,, u_\nu)$ *Leray weak solutions to the Navier-Stokes :*

$$\frac{1}{2}\|u_\nu(t)\|^2_{L^2(\Omega)} + \nu \int_0^t \|\nabla_x u_\nu(t)\|^2_{L^2(\Omega)} dt \leq \frac{1}{2}\|u_\nu(0)\|^2_{L^2(\Omega)} \tag{1}$$

uniformly in $\nu \to 0$. *Assume that their vorticity* $\omega_\nu = \nabla^\perp \cdot u_\nu$ *satisfies*

$$\limsup_{\nu\to 0}\Big(- \int_0^T \int_{\partial\Omega} \nu\omega_\nu(t,\sigma) u(t,\sigma)\cdot\tau(\sigma) d\sigma dt\Big) = 0, \tag{2}$$

then any $\overline{u}_\nu$, *which is a* **weak**$-*$ **limit** *in* $L^\infty([0, T]; L^2(\Omega))$ *of a subsequence* u_{ν_j} *as* $\nu_j \to 0$, *satisfies the stability estimate (and convergence for* $\overline{u_\nu(0)} - u(0) = 0$).

$$\|\overline{u_\nu(t)} - u(t)\|^2_{L^2(\Omega)} \leq e^{2t\|\nabla u\|_{L^\infty([0,T]\times\Omega)}}\|\overline{u_\nu(0)} - u(0)\|^2_{L^2(\Omega)}. \tag{3}$$

Hence our purpose is the most possibly direct proof of (2) for short time assuming that boundary of the domain and initial value of the solution are analytic. Object of the Theorem 2 (main):

Theorem

2 **Main Theorem**

Let $u_0(x)$ be an initial data that is analytic up to the boundary $\partial\Omega$ and vanishes on the boundary. Then, there is a positive time T, independent of ν, so that the unique solutions $u_\nu(t)$ to the Navier-Stokes problem satisfies the estimate

$$\lim_{\nu\to 0} \sqrt{\nu}\|\omega_\nu\|_{L^\infty([0,T]\times\partial\Omega))} < \infty. \tag{4}$$

Observe that with (1) one has

$$\nu \int_0^T \|\omega_\nu(t)\|^2_{L^2(\Omega)} dt \leq C_0 \tag{5}$$

and the theorem 2 gives

$$\nu \int_0^T \|\omega_\nu(t)\|^2_{L^2(\partial\Omega)} dt \leq C_0 \tag{6}$$

While with Kato-duality type argument (depending on the regularity of u_τ)

$$\|\omega_\nu(t)\|_{L^2(\partial\Omega\times(0,T))} = o(\nu) \tag{7}$$

would be enough to have (2) .

(2) a bit sharper than (5). But not so much!!! However the difference plays a crucial role in wall turbulence.

Since analytic solutions of the Euler equations are considered the estimate (6) is even an overkill for the proof of (4). Up to now we have not been able to provide a proof of convergence with a weaker estimate than $O(\sqrt{\nu})$ which is reminiscent of the Prandlt scaling. And in fact Prandlt expansion was used for a first result in the present direction by C. Wang and Y. Wang J. Math. Fluid Mech. (2020).

The proof is build on the extension to any domain with analytic curved boundary of the following recent tools concerning the half space.

1. C. R. Anderson Vorticity boundary conditions and boundary vorticity generation for two-dimensional viscous incompressible flows. J. Comput. Phys. 1989 and Y. Maekawa, On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. Comm. Pure Appl. Math. (2014).
2. T.T. Nguyen and T.T. Nguyen. The inviscid limit of Navier-Stokes equations for analytic data on the half-space. Arch. Ration.Mech. Anal., 2018.
3. The release of the analyticity hypothesis away from the boundary I. Kukavica, V Vicol and F. Wang, Arch. Ration. Mech. Anal. (2020),.

On $\partial\Omega$ one has:

$$0 = \tau \cdot \partial_t u = \tau \cdot \nabla^\perp \Delta^{-1} \partial_t \omega = \partial_n[\Delta^{-1}(\nu\Delta\omega - u \cdot \nabla\omega)] \qquad (8)$$

With Dirichlet Neumann operator and $\vec{n}$ interior normal

$$\omega^* = \omega \quad \text{on } \partial\Omega\,, -\Delta\omega^* = 0, \text{in} \quad \Omega \quad DN(\omega) = -\partial_n\omega^*, \qquad \text{on} \quad \partial\Omega,$$
$$\partial_n[\Delta^{-1}\Delta\omega] = \partial_n[\Delta^{-1}\Delta(\omega - \omega^*)] = (\partial_n + DN)\omega\,. \qquad (9)$$

$$(\partial_n\omega_\nu + DN\omega_\nu) = \frac{1}{\nu}\partial_{\vec{n}}\Delta^{-1}(u \cdot \nabla\omega_\nu)\,,$$
$$\partial_t\omega_\nu + u_\nu \cdot \nabla\omega_\nu - \nu\Delta\omega_\nu = 0\,. \qquad (10)$$

Remark

With the standard energy estimates :

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |\omega_\nu(x,t)|^2 dx + \nu\int_\Omega |\nabla_\nu \omega_\nu(x,t)|^2 dx = \int_{\partial\Omega} DN\omega_\nu\omega_\nu d\sigma + \frac{1}{\nu}\partial_{\vec{n}}\Delta^{-1}(u_\nu\cdot\nabla\omega_\nu)d\sigma \quad (11)$$

indicating that the problem is ill posed even for $\nu > 0$ in any Sobolev space. However it is well posed in space of analytic functions. And ω_ν is analytic in $(t>0, X+iY, X\in\Omega\times Y\in\mathbb{R}^2)$ while the solution of the Euler equation with analytic initial data is also analytic for

$$t\geq 0, X+iY, X\in\Omega\times|Y|\leq Ce^{-Ce^{Ct}}.$$

$$\forall x \in \mathbb{R}^2\,, d(x,\partial\Omega) = \inf_{y\in\partial\Omega} |x-y|\,, V_\delta = \{x \in \mathbb{R}^2, d(x,\partial\Omega) < \delta\}$$

Analytic curve : $\quad \partial\Omega = \{\theta \in \mathbb{T} = \mathbb{R}/(\mathbb{Z}L) \mapsto x(\theta) = (x_1(\theta), x_2(\theta))\}$

Tangent, interior normal curvature and distance:

$$\begin{aligned}
\vec\tau(\theta) = \vec\tau(x(\theta)) = &(x_1'(\theta), x_2'(\theta)), \quad \vec n(\theta) = \vec n(x(\theta)) = (-x_2'(\theta), x_1'(\theta))\\
&\text{with} \quad |x'(\theta)|^2 = (x_1'(\theta))^2 + (x_2'(\theta))^2 = 1.\\
&\gamma(\theta) = x_1''(\theta)x_2'(\theta) - x_1'(\theta)x_2''(\theta),\\
&d(x,\partial\Omega) = \inf_{y\in\partial\Omega} |x-y| = |x - x(\theta)|\,.
\end{aligned} \tag{12}$$

There exists a $\delta > 0$ such that the mapping

$$(\theta, z) \in \{\mathbb{T} = \mathbb{R}/(\mathbb{Z}L) \times |z| < \delta_0\} \mapsto x(\theta) + z\vec n(\theta) \tag{13}$$

is an analytic isomorphism on V_δ and one has $d(x,\partial\Omega) = |z|\,.$

Under the scaling diffusion equation

$$(\tilde{t},\tilde{\theta},\tilde{z}) = (\lambda^2 t, \lambda\theta, \lambda z) \tag{14}$$

with $\delta_0 = \lambda\delta$ the above representation is changed into the isomorphism

$$(\tilde{\theta},\tilde{z}) \in \{\mathbb{T} = \mathbb{R}/(\mathbb{Z}\tilde{L}) \times |\tilde{z}| < \delta_0\} \mapsto \tilde{x}(\tilde{\theta}) + \tilde{z}\vec{n}(\tilde{\theta}) \mapsto V_\delta\,.$$

Moreover with the above scaling one has

$$\gamma(\tilde{\theta}) = \lambda^3\gamma(\theta)\,. \tag{15}$$

Below, unless it is compulsory, and in the absence of risk of confusion, the sign $\tilde{.}$ will be omitted for functions depending of $(\tilde{t},\tilde{\theta},\tilde{z}) = (\lambda^2 t, \lambda\theta, \lambda z)$.

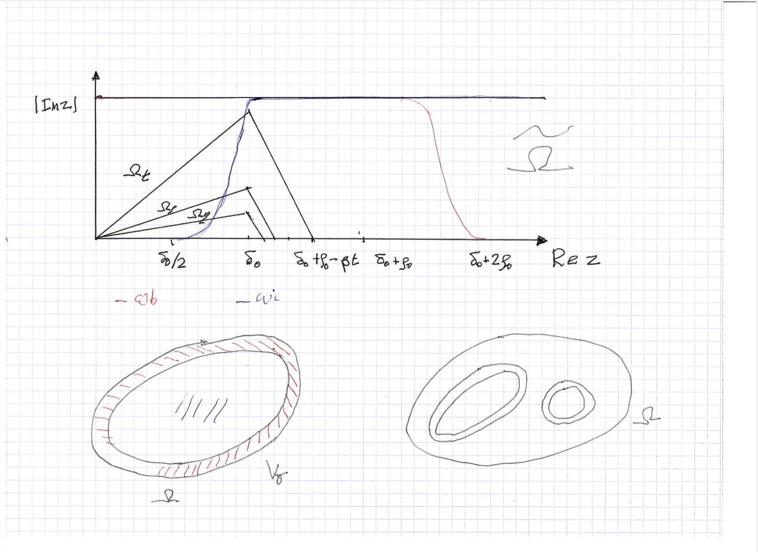
$$\phi^b(x) = \begin{cases} 1, & \text{if} \quad \lambda d(x,\partial\Omega) \le \delta_0 + \rho_0 \\ 0, & \text{if} \quad \lambda d(x,\partial\Omega) \ge \delta_0 + 2\rho_0 \end{cases}$$

$$\phi^i(x) = \begin{cases} 0, & \text{if} \quad \frac{\delta_0}{2} < \lambda d(x,\partial\Omega) \\ 1 & \text{if} \quad \lambda d(x,\partial\Omega) \ge \delta_0 \end{cases} \tag{16}$$

$$\omega_\nu \simeq \phi^b\omega_\nu + \phi^i\omega_\nu = \omega_\nu^b + \omega_\nu^i$$

$$u^b = \nabla^\perp\Delta^{-1}\omega^b, \qquad u^i = \nabla^\perp\Delta^{-1}\omega^i.$$

ω_ν^i is extended by 0 over $\mathbb{R}^2$ and will be estimated in term of the H^3 Sobolev norm of ω_ν^b restricted to the domain $\frac{\delta_0}{2} < \lambda d(x,\partial\Omega) < \delta_0$. Then the estimates of ω_ν^b relies on analytic estimates and H^3 Sobolev norm of ω_ν^i restricted to the domain $\delta_0 + \rho_0 < \lambda d(x,\partial\Omega)$.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From the Anderson-Mayekawa boundary condition

$$(\partial_n \omega_\nu + DN\omega_\nu) = \frac{1}{\nu}\partial_{\vec{n}}\Delta^{-1}(u \cdot \nabla\omega_\nu)$$
$$\partial_t \omega_\nu + u_\nu \cdot \nabla\omega_\nu - \nu\Delta\omega_\nu = 0 \qquad (17)$$

one deduces for $\omega_\nu^b = \phi^b \omega_\nu^b$

$$\partial_t \omega_\nu^b + u_\nu^b \cdot \nabla\omega_\nu^b - \nu\Delta\omega_\nu^b = K_1$$
$$\text{with} \quad K_1 = \nu(2\nabla_x(\nabla_x \phi^b \omega_\nu) - \Delta\phi^b\omega)$$
$$+ (u \cdot \nabla_x \phi^b)\omega_\nu) + ((1-\phi^b)u_\nu) \cdot \nabla\omega_\nu^b \qquad (18)$$
$$\nu(\partial_n \omega_\nu^b + DN\omega_\nu^b) = \partial_{\vec{n}}\Delta^{-1}(u^b \cdot \nabla\omega_\nu^b) + K_2$$
$$\text{with} \quad K_2 = \partial_{\vec{n}}\Delta^{-1}\nabla \cdot (u_\nu\omega_\nu - u_\nu^b\omega_\nu^b)$$

Then rescaled geodesic coordinates are used for ω_ν^b as a function defined in

$$\{\mathbb{T} = \mathbb{R}/(\mathbb{Z}\tilde{L}) \times \mathbb{R}_z^+\}$$

with support in $0 < z < \delta_0 + 2\rho_0$ In such representation :

$$\begin{aligned}
&\Delta \mapsto \lambda^2(\partial_z^2 + \partial_\theta^2 + \lambda^2 m(z,\theta)\partial_\theta^2 + R_\Delta) \\
&\text{with} R_\Delta = \frac{\gamma}{1+z\gamma}\partial_z - \frac{z\gamma'}{(1+z\gamma)^3}\partial_\theta \qquad m(z,\theta) = -\frac{2z\gamma + (z\gamma)^2}{(1+z\gamma)^2)^2} \qquad (19) \\
&DN \mapsto |\partial_\theta| + B \quad \text{with} \quad B \in \mathcal{L}(L^2(\mathbb{R}/(\mathbb{Z}L)) .
\end{aligned}$$

Eventually changing t into $\lambda^2 t$ one has in the rescaled variables $(\theta, z) \in \mathbb{R}/(\mathbb{Z}\tilde{L}) \times \mathbb{R}^+_z$:

$$\begin{aligned} &\partial_t \omega^b_\nu - \nu\Delta\omega^b_\nu = -\nu\lambda^2(m(z,\theta)\partial^2_\theta\omega^b) - \lambda^{-2}u^b_\nu \cdot \nabla\omega^b_\nu + \overline{K_1(\lambda)} \\ &\nu(\partial_n\omega^b_\nu + |\partial_\theta|\omega^b_\nu) = \lambda^{-1}[\partial_z\Delta^{-1}(u^b \cdot \nabla\omega^b_\nu)] - \nu B(\omega_\nu) + \overline{K_2(\lambda)}\,. \end{aligned} \tag{20}$$

The role of λ is to "flatten " the curvature near the boundary. In the change of variables $\theta \mapsto \lambda\theta$ the curvature is changed into $\lambda^3\gamma(\lambda\theta)$ this makes appear the coefficient λ^2 in front of $\lambda^2 m(z,\theta)\partial^2_\theta\omega^b$ which then can be dominated by the laplacian. This goes very well with the observation of vortices generated in the fluid by curved boundary "Gortler Vortices"

Introduce the complexification of Ω near the $\partial\Omega$ as

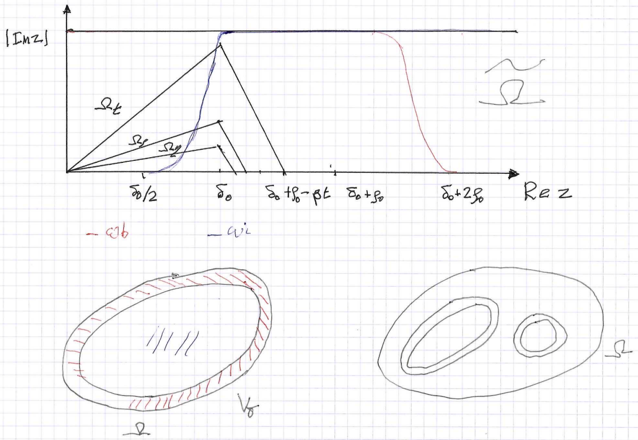
$$\text{With}\quad 0 < \frac{\delta_0}{2} < \delta_0 < \delta_0 + \rho_0 < \delta_0 + 2\rho_0 < \delta_0 + 3\rho_0 < \delta \quad \Omega_\rho \subset \Omega_{\delta_0}$$

$$\Omega_\rho = \{z \in \mathbb{C} : 0 \le \Re z \le \delta_0, |\Im z| \le \rho \Re z\}$$
$$\cup \{z \in \mathbb{C} : \delta_0 \le \Re z \le \delta_0 + \rho, |\Im z| \le \delta_0 + \rho - \Re z\}$$

With $\alpha \in \mathbb{Z}$ Fourier modes for the decomposition in θ.

$$\|f\|_{L^1_\rho} = \sup_{0\le\eta<\rho} \|f\|_{L^1(\partial\Omega_\eta)}, \qquad \|f\|_{L^\infty_\rho} = \sup_{0\le\eta<\rho} \|f\|_{L^\infty(\partial\Omega_\eta)}$$

$$\|f\|_{\mathcal{L}^1_\rho} = \sum_{\alpha\in\mathbb{Z}} \|e^{\epsilon_0(\delta_0+\rho-\Re z)|\alpha|} f_\alpha\|_{L^1_\rho},$$
$$\|f\|_{\mathcal{L}^\infty_\rho} = \sum_{\alpha\in\mathbb{Z}} \|e^{\epsilon_0(\delta_0+\rho-\Re z)|\alpha|} f_\alpha\|_{L^\infty_\rho}, \tag{21}$$
$$\|f\|_{\mathcal{W}^{k,p}_\rho} = \sum_{i+j\le k} \|\partial_\theta^i (z\partial_z)^j f\|_{\mathcal{L}^p_\rho}$$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Introduce a compound norm made of the analytic norm of ω_ν^b evaluated in geodesic rescaled variable and of a Sobolev norm of ω_ν^i, extension to $\mathbb{R}^2$ of the truncated solution and write with $\delta_0, \rho_0, \lambda$ small enough and $\zeta \in (0,1)$

$$A(\beta,\lambda,\omega)(t) = \sup_{0<\rho<\rho_0-\beta\lambda^2 t}\Big\{\|\omega^b(t)\|_{\mathcal{W}^{1,1}_\rho} + \|\omega^b(t)\|_{\mathcal{W}^{2,1}_\rho}(\rho_0-\rho-\lambda^2\beta t)^\zeta\Big\}\Big] \tag{22}$$

Theorem For the compound norms

$$\mathcal{CA}(\omega) = \sup_{0<\beta t<\lambda^2\rho_0}\Big[A(\beta,\lambda,\omega)(t) \quad + \|\omega(t)\|_{H^4(\{\lambda d(x,\partial\Omega)\geq\delta_0/2\})}\Big] \tag{23}$$

one has the estimate:

$$\begin{aligned}\mathcal{CA}(\beta,\lambda,\omega_\nu,) \leq C[\|\omega(0)\|_{\mathcal{W}^{2,1}_\rho} + \|\omega(0)\|_{H^4(\{\lambda d(x,\partial\Omega)\geq\delta_0/2\})}]\\ +D\lambda^{-2}\beta^{-1}(\mathcal{CA}(\beta\lambda,\omega_\nu,))^2\,.\end{aligned} \tag{24}$$

In Fourier decomposition, for analytic variables according to Nguyen[2]. the Duhamel formula becomes:

$$(e^{\nu tS}f)_\alpha(z)=\int_0^\infty G_\alpha(t,y;z)f_\alpha(y)\,dy,\ \ (\Gamma(\nu t)g)_\alpha(z)=G_\alpha(t,0;z)g_\alpha, \tag{25}$$

with:

$$G_\alpha(t,y;z)=H_\alpha(t,y;z)+R_\alpha(t,y;z), \tag{26}$$

where

$$H_\alpha(t,y;z)=\frac{1}{\sqrt{\nu t}}\Big(e^{-\frac{|y-z|^2}{4\nu t}}+e^{-\frac{|y+z|^2}{4\nu t}}\Big)e^{-\alpha^2\nu t},$$

$$|\partial_z^k R_\alpha(t,y;z)|\lesssim \mu_f^{k+1}e^{-\theta_0\mu_f|y+z|}+(\nu t)^{-\frac{k+1}{2}}e^{-\theta_0\frac{|y+z|^2}{\nu t}}e^{-\frac{1}{8}\alpha^2\nu t},$$

Inserts for f and g the right hand side of

$$\begin{aligned}\partial_t \omega_\nu^b - \nu\Delta\omega_\nu^b &= -\nu\lambda^2(m(z,\theta)\partial_\theta^2\omega^b) - \lambda^{-2}u_\nu^b\cdot\nabla\omega_\nu^b + \overline{K_1(\lambda)}\\ \nu(\partial_n\omega_\nu^b + |\partial_\theta|\omega_\nu^b) &= \lambda^{-1}[\partial_z\Delta^{-1}(u^b\cdot\nabla\omega_\nu^b)] - \nu B(\omega_\nu) + \overline{K_2(\lambda)}\,.\end{aligned} \tag{27}$$

The proof is completed with the following observations:

$$\begin{aligned}&\bullet \quad \nu\lambda^2\|\int_0^t e^{\nu(t-t')S} m(z,\theta)\partial_\theta^2\omega^b\, dt'\|_{\mathcal{W}_\rho^{k,1}}\\ &\lesssim \nu\lambda^2\int_0^t \|\partial_\theta^2\omega(t')\|_{\mathcal{W}_\rho^{k,1}}\, dt' + \|\omega\|_{H^{k+1}(\lambda d(x,\partial\Omega)\ge\delta_0+\delta)}\end{aligned} \tag{28}$$

This term will be absorbed by the left hand side of the final estimate provided λ which is now fixed is chosen small enough.

- The support of $\overline{K_1}$ and $\overline{K_2}$ are contained in the region $\delta_0 + 2\rho_0 \le z \le \delta_0 + 3\rho_0$ hence the "analytic norm " of the solution of

$$\partial_t\tilde\eta - \nu\Delta\tilde\eta = \overline{K_1} \quad (\partial_{\vec n}\tilde\eta + |\partial_\theta|\tilde\eta) = \overline{K_2} \tag{29}$$

involving the weight $e^{-\frac{|y-z|^2}{4\nu t}}$ with $|y-z| > \rho_0$ is bounded by $C\|\omega_0\|_{H^4(\{\lambda d(x,\partial\Omega)\ge\delta_0/2\})}\,.$

- Remains the non linear terms ie the solution of

$$\begin{aligned} &\partial_t\tilde{\omega} - \nu\Delta\tilde{\omega} = -\lambda^{-2}(u_\nu^b \cdot \nabla\omega_\nu^b) \\ &(\partial_n\tilde{\omega} + |\partial_\theta|\tilde{\omega}) = \lambda^{-1}\partial_{\vec{n}}\Delta^{-1}(u^b \cdot \nabla\omega_\nu^b) \end{aligned} \tag{30}$$

$\tilde{\omega}$ is defined for $z \in \mathbb{R}_+$ with support in $0 < z < \delta_0 + 3\rho_0$ and is analytic norm for $z < \delta_0 + 2\rho_0$. Estimation of $A(\beta,\lambda,\omega)(t)$ follows using

$$\begin{aligned} &\|fg\|_{\mathcal{L}^1_\rho} \leq \|f\|_{\mathcal{L}^\infty_\rho}\|g\|_{\mathcal{L}^1_\rho} \\ &\|\partial_\theta f\|_{\mathcal{L}^1_\rho} + \|z\partial_z f\|_{\mathcal{L}^1_\rho} \lesssim \frac{1}{\rho' - \rho}\|f\|_{\mathcal{L}^1_{\rho'}} . \end{aligned} \tag{31}$$

For instance with $\rho' = \frac{1}{2}(\rho + \rho_0 - \beta t)$ and $\zeta \in (0,1)$ one has

$$\begin{aligned} &\|u \cdot \nabla\omega^b\|_{\mathcal{W}^{1,1}_\rho} \lesssim \|\omega\|_{\mathcal{W}^{1,1}_\rho}\|\omega\|_{\mathcal{W}^{1,2}_\rho} + \|\omega\|^2_{H^2(\{\lambda d(x,\partial\Omega)\geq\delta_0\})} \\ &\lesssim A(\beta)^2(\rho_0 - \rho - \beta\lambda^2 t)^{-\tilde{\zeta}}. \end{aligned} \tag{32}$$

Eventually $\omega^i_\nu = \phi^i(x)\omega_\nu$ is compactly supported in Ω and solution of the equation

$$\partial_t \omega^i_\nu + u^i_\nu \cdot \nabla \omega^i_\nu - \nu \Delta \omega^i_\nu = J \tag{33}$$

with $\mathrm{support} \quad J \subset \{\frac{\delta_0}{2} < d(\lambda x, \partial\Omega) < \delta_0\}$ where ω^i_ν coincide with ω^b therefore by standard Sobolev estimates one has:

$$\frac{d}{dt}\|\omega^i\|^2_{H^k} \lesssim \|\omega^i\|^2_{H^k}(\|\omega^b\|_{\mathcal{W}^{k,1}_\rho} + \|\omega^i\|_{H^k}) \tag{34}$$

Hence providing the road map for the compound theorem.

Start from the relation

$$\|\omega_\nu^b\|_{L^\infty(\partial\Omega)} \lesssim \|\partial_{\tilde z}\omega_\nu^b\|_{\mathcal{L}^1_\rho} + \|\omega_\nu(t)\|_{H^2(\{\lambda d(x,\partial\Omega)\geq\delta_0/2\})}.$$

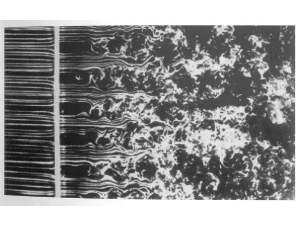
Differentiate with respect to z the Duhamel formula, with f replaced by ω_ν^b makes appears a term $O(\nu^{-\frac{1}{2}})$. Then insert the estimate (24) of the compound theorem to conclude:

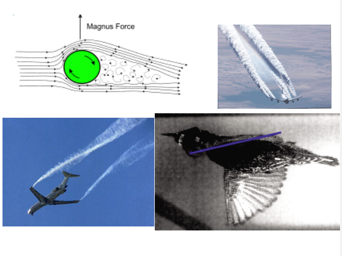
$$\sqrt{\nu}\|\omega_\nu(t,x)\|_{L^\infty(t\in(0,\frac{\lambda^2\rho_0}{\beta}))\times\partial\Omega)} \leq \mathit{Constante}\,. \qquad (35)$$

- The fact that the weak limit $\overline{u_\nu}$ would not be a solution of the Euler equation seems to in agreement with numerical and physical observations concerning the boundary effect, with the anomalous energy dissipation and with the force applied to these wall by the fluid (the d'Alembert paradox).
- Whenever the Prandlt equation have a solution which do describes the behavior of u_ν near the wall then convergence holds.
- However it is known that even with analytic initial data these solution may exhibit singularities after a finite time (W. E and B. Enquist CPAM 1997) .

- Moreover without analyticity one can construct examples where the Prandlt equations do have a solution which does not match (at least in L^∞) the behavior of u_ν for $\nu \to 0$ (E. Grenier and T. Nguyen, L^∞ instability of Prandtl layers. Ann. PDE (2019)).
- With shear flow and rotating flows it is easy to construct solutions that weakly converge to the solution of the Euler equation (C.B. , E.S. Titi and E. Wiedemann, C.R. Acad. Sci. Paris, 350 (2012)), underlying the fact that the issue depends in sense more on the geometry than on the regularity.

- The present construction provides a result valid for any geometry, for short time but based on the Prandlt scaling. In fact the time is very short. First λ has to be chosen small enough to flatten then β has to be chosen large enough to make the iteration work.
- Estimate of $\nu^\alpha \omega_\nu$weakly $\rightarrow 0$ on $\partial\Omega$ with $\alpha < 1$ would be enough. Up to now we have not been able to produce such result. May be one would try to match such point of view with results on the Gevrey stability of Prandlt equations.(D. Gérard-Varet, Y. Maekawa, and N. Masmoudi, Duke Math. J. 167 (2018) .
- Eventually in real experiment wall effect appear in stationary regime. This seems much more difficult to approach than short time results related to situation where both $\nu \rightarrow 0$ while $t \rightarrow \infty$ or genuine statistical theory of turbulence.





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