Morrey's problem in classes of homogeneous integrands

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Talk based on joint work with

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• *Automatic quasiconvexity of homogeneous isotropic rank-one convex integrands*, with André Guerra
ARMA **245** (2022), no. 1, 479–500.

• *Restricted quasiconvexity of the Burkholder functional*,
with Kari Astala, Daniel Faraco, André Guerra, Aleksis Koski
In preparation 2022.

Morrey's problem (1952):
Does rank-one convexity imply quasiconvexity?

Šverák (1992):
An example of a quartic polynomial on $\mathbb{R}^{N\times n}$ that is rank-one convex but not quasiconvex when the dimensions $N \geq 3$, $n \geq 2$.

Grabovsky (2016):
An example of a nonnegative 2-homogeneous integrand on $\mathbb{R}^{8\times 2}$ that is rank-one convex but not quasiconvex.

Morrey's problem remains open in dimensions $N = 2$, $n \geq 2$.

Outline

- The convexity notions and background
- Examples
- Results and some consequences
- Comments on proofs

The convexity notions and background

A motivation: Given an integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$. Let $1 \leq p < \infty$ and assume F is continuous and of natural p-growth

$$\bigl|F(z)\bigr| \leq L\bigl(|z|^p + 1\bigr) \quad \forall\, z$$

Given a bounded open subset Ω of $\mathbb{R}^n$ and $g \in \mathrm{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^N)$.

Minimize

$$\int_\Omega F(\nabla u(x))\,\mathrm{d}x$$

over $u \in \mathrm{W}^{1,p}_g \equiv \mathrm{W}^{1,p}_g(\Omega, \mathbb{R}^N)$.

Only systematic approach to existence is the direct method

Coercivity

Assume $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ continuous and of natural p-growth.

TFAE: (Chen & JK 2017, Gmeineder & JK 2019)

- All minimizing sequences are bounded in $\mathrm{W}^{1,p}_{g}$
- There exists $c_1 > 0$ such that

$$\int_\Omega F(\nabla u(x))\,\mathrm{d}x \geq c_1 \int_\Omega |\nabla u(x)|^p\,\mathrm{d}x - \frac{1}{c_1} \quad \forall\, u \in \mathrm{W}^{1,p}_{g}$$

- There exist $c_2 > 0$ and $z_0 \in \mathbb{R}^{N\times n}$ such that

$F - c_2|\cdot|^p$ is quasiconvex at z_0.

Sequential weak lower semicontinuity

Assume $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ continuous and of natural p-growth.

TFAE: (Morrey 1952, Meyers 1965, Fusco 1981, ...)

- If u_j, $u \in \mathrm{W}^{1,p}_{g}(\Omega, \mathbb{R}^N)$ and $u_j \rightharpoonup u$ in $\mathrm{W}^{1,p}$, then

$$\liminf_{j\to\infty} \int_\Omega F(\nabla u_j(x))\,\mathrm{d}x \geq \int_\Omega F(\nabla u(x))\,\mathrm{d}x$$

- F is quasiconvex

Quasiconvexity (Morrey 1952)

- An integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex at $z \in \mathbb{R}^{N\times n}$ if

$$\int_{\mathbb{X}} F(z+\nabla\phi(x))\,\mathrm{d}x \geq F(z) \quad \forall\,\phi \in \mathrm{W}_0^{1,\infty}(\mathbb{X},\mathbb{R}^N) \tag{1}$$

where $\mathbb{X} \equiv \left(-\frac{1}{2},\frac{1}{2}\right)^n$ is the open unit cube in $\mathbb{R}^n$.

- F is quasiconvex if it is quasiconvex at all $z \in \mathbb{R}^{N\times n}$.

Note: When F has natural p-growth and is quasiconvex at z, then (1) holds for all $\phi \in \mathrm{W}_0^{1,p}(\mathbb{X},\mathbb{R}^N)$. In fact, it then holds for all homogeneous $\mathrm{W}^{1,p}$ gradient Young measures (Acerbi & Fusco 1984, Ball & Zhang 1990)

Rank-one convexity

• An integrand $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is rank-one convex if it is convex in all rank-one directions:

$$\mathbb{R} \ni t \mapsto F(z+tw) \text{ is convex}$$

for all z, $w \in \mathbb{R}^{N\times n}$ with $\mathrm{rank}(w) = 1$.

Legendre-Hadamard condition: When F is C^2, then F is rank-one convex iff

$$F''(z)[w,w] \geq 0$$

holds for all z, $w \in \mathbb{R}^{N\times n}$ with $\mathrm{rank}(w) = 1$.

$$QC \Rightarrow RC$$

Quasiaffine = rank-one affine & Definition of polyconvexity

- $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiaffine (rank-one affine) if $\pm F$ are QC (RC).

TFAE:

- F is rank-one affine
- F is quasiaffine
- $F(z)$ is an affine function of z and its minors
- F is a Null Lagrangian

Ball 1977: $F\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is polyconvex if $F(z)$ is a convex function of z and its minors.

- F is polyconvex at z if there exists a Null Lagrangian A such that

$$F \geq A \text{ and equality holds at } z$$

The Dacorogna–Marcellini integrand (1987)

$$\mathrm{DM}_\gamma(z) \equiv |z|^2\big(|z|^2 + \gamma \mathrm{det} z\big)$$

where $z \in \mathbb{R}^{2\times 2}$, $|z| \equiv \sqrt{\mathrm{trace}\ (z^T z)}$ and $\gamma \in \mathbb{R}$ is a parameter.

- DM_γ is a homogeneous polynomial of degree 4
- DM_γ is isotropic: $\mathrm{DM}_\gamma(QzR) = \mathrm{DM}_\gamma(z)$ for all $z \in \mathbb{R}^{2\times 2}$ and $Q,\ R \in \mathrm{SO}(2)$.

Theorem: (Dacorogna & Marcellini '87, Alibert & Dacorogna '92)

(i) DM_γ is convex iff $|\gamma| \leq \frac{4}{3}\sqrt{2}$

(ii) DM_γ is polyconvex iff $|\gamma| \leq 2$.

(iii) DM_γ is rank-one convex iff $|\gamma| \leq \frac{4}{\sqrt{3}}$

(iv) DM_γ is quasiconvex iff $|\gamma| \leq 2 + \varepsilon$ for some $\varepsilon \in (0, \frac{4}{\sqrt{3}} - 2]$

The Burkholder integrand (1984), Iwaniec (2002)

$$\mathbf{B}_p(z) \equiv \mathbf{B}_{p,n}(z) \equiv \|z\|^{p-n}\left(\left|1-\frac{n}{p}\right|\|z\|^n - \det z\right)$$

where $z \in \mathbb{R}^{n\times n}$ and $\|z\| \equiv \max_{|x|=1}|zx|$.

- $\mathbf{B}_p$ is positively p-homogeneous: $\mathbf{B}_p(tz) = t^p\mathbf{B}_p(z)$ for all $z \in \mathbb{R}^{n\times n}$ and $t > 0$.
- $\mathbf{B}_p$ is isotropic: $\mathbf{B}_p(QzR) = \mathbf{B}_p(z)$ for all $z \in \mathbb{R}^{n\times n}$, Q, $R \in \mathrm{SO}(n)$.
- $\mathrm{B}_n(z) = -\det z$

Theorem: (Iwaniec 2002)

$\mathbf{B}_p$ is rank-one convex iff $p \geq \frac{n}{2}$.

A connection to quasiregular maps

Let Ω be an open subset of $\mathbb{R}^n$ and $K \geq 1$. Then $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is weakly K-quasiregular if

$$\|\nabla u\|^n \leq K \det \nabla u$$

holds almost everywhere in Ω.

Theorem (Iwaniec 2002): If B_p is quasiconvex at 0 for all $p > \frac{n}{2}$, then for $K > 1$ and $s > \frac{nK}{K+1}$ any weakly K-quasiregular maps in $\mathrm{W}^{1,s}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ belongs to $\mathrm{W}^{1,t}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for each $t < \frac{nK}{K-1}$.

Theorem (Astala 1994): The Sobolev range for weakly K-quasiregular maps in dimension $n = 2$ is $\left(\frac{2K}{K+1}, \frac{2K}{K-1}\right)$.

The Iwaniec conjecture

The Burkholder integrand in dimension $n = 2$

$$\mathrm{B}_p(z) \equiv \|z\|^{p-2}\left(\left|1-\tfrac{2}{p}\right|\|z\|^2 - \det z\right), \quad z \in \mathbb{R}^{2\times 2}$$

is quasiconvex for all $p \geq 1$.

Guerra (2018): B_p is quasiconvex iff it is quasiconvex at 0.

Dacorogna (1994): B_1 is convex.

More integrands from Iwaniec (2002)

$$\mathbb{B}_{p,\gamma}(z) \equiv \big(\gamma|z^-| - |z^+|\big)\big(|z^+| + |z^-|\big)^{p-1}, \quad z \in \mathbb{R}^{2\times 2}, \tag{2}$$

where $z^{\pm} \equiv \frac{1}{2}\big(z \pm \mathrm{cof} z\big)$ and $p \geq 1$, $\gamma > 0$ are parameters.

- $\mathbb{B}_{p,\gamma}$ is rank-one convex iff $p > 1$ and $\gamma \geq \max\{p, \frac{p}{p-1}\} - 1$.

In particular

$$\mathbb{B}_{p,\infty}(z) \equiv |z^-|\big(|z^+| + |z^-|\big)^{p-1}, \quad z \in \mathbb{R}^{2\times 2},$$

is rank-one convex.

The integrands

$$\mathrm{L}_{\pm}(z) \equiv \frac{1}{n}\|z\|^n \pm \log\|z\| \, \det z, \quad z \in \mathbb{R}^{n\times n} \tag{3}$$

are rank-one convex.

Theorem: (Guerra & JK 2021)

Let $F\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ be rank-one convex. Assume

(i) F is isotropic: $F(QzR) = F(z)$ for $z \in \mathbb{R}^{n\times n}$, Q, $R \in \mathrm{SO}(n)$.

(ii) F is positively p-homogeneous and $p \geq n$: $F(tz) = t^p F(z)$ for $z \in \mathbb{R}^{n\times n}$, $t > 0$.

(iii) F is nonnegative: $F(z) \geq 0$ for $z \in \mathbb{R}^{n\times n}$.

Then F is polyconvex at each $z \in \mathrm{CO}(n)$, where

$$\mathrm{CO}(n) \equiv \{tQ : \ Q \in \mathrm{O}(n), \ t \in \mathbb{R}\}.$$

Proposition: (Guerra & JK 2021)

- For each $p \geq 2$, the integrand $\mathbf{B}_p^+\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is polyconvex.
- For each $p > 3$, the integrand $\mathbf{B}_p^+\colon \mathbb{R}^{3\times 3} \to \mathbb{R}$ is not polyconvex.

Note: $\mathbf{B}_p^+(z) \equiv \max\{\mathbf{B}_p(z), 0\}$

Theorem: (Guerra & JK 2021)

Let $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ be rank-one convex and satisfy

(i) F is isotropic: $F(QzR) = F(z)$ for $z \in \mathbb{R}^{2\times 2}$, Q, $R \in \mathrm{SO}(2)$.

(ii) F is positively p-homogeneous and $1 < p < 2$:
$F(tz) = t^p F(z)$ for $z \in \mathbb{R}^{2\times 2}$, $t > 0$.

(iii) F is nonnegative: $F(z) \geq 0$ for $z \in \mathbb{R}^{2\times 2}$.

If $\mathbf{B}_p$ is quasiconvex in dimension $n = 2$, then F is quasiconvex at each $z \in \mathrm{CO}(2)$, where

$$\mathrm{CO}(2) \equiv \left\{ tQ : \, Q \in \mathrm{O}(2), \, t \in \mathbb{R} \right\}.$$

Burkholder and weakly quasiregular maps in the plane

Recall

- $u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is weakly K-quasiregular if

$$u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \text{ and } \|\nabla u\|^2 \leq K \det \nabla u$$

- The Burkholder integrand in dimension $n = 2$ is

$$\mathrm{B}_p(z) \equiv \|z\|^{p-2}\left(\left|1 - \tfrac{2}{p}\right|\|z\|^2 - \det z\right), \quad z \in \mathbb{R}^{2\times 2}$$

Observation: If $v \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$, then for $p > 2$:

$$\mathbf{B}_p(\nabla v) \leq 0 \quad \Leftrightarrow \quad \|\nabla v\|^2 \leq \tfrac{p}{p-2} \det \nabla v$$

Note $K = \frac{p}{p-2}$ means $p = \frac{2K}{K-1}$.

Theorem: (Astala, Faraco, Guerra, Koski & JK 2022)

Let $p > 2$. Then the integrand

$$F(z) \equiv \begin{cases} \mathsf{B}_p(z) & \text{if } \mathsf{B}_p(z) \leq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

is closed $\mathrm{W}^{1,p}$ quasiconvex.

Consequently, B_p is in particular quasiconvex in the following restricted sense:

If $z \in \mathbb{R}^{2\times 2}$, $\phi \in \mathrm{W}_0^{1,p}(\mathbb{X}, \mathbb{R}^2)$, then

$$\mathsf{B}_p(z + \nabla\phi) \leq 0 \quad \Rightarrow \quad \mathsf{B}_p(z) \leq \int_{\mathbb{X}} \mathsf{B}_p(z + \nabla\phi(x))\, \mathrm{d}x.$$

Notation: $\mathbb{X} \equiv \left(-\frac{1}{2}, \frac{1}{2}\right)^2$.

Consequence

Assume $u\colon B_1(0) \to \mathbb{R}^2$ is weakly K-quasiregular and $u(x) = zx$ for $x \in \partial B_1(0)$.

Then $u \in \mathrm{W}^{1,p}(B_1(0), \mathbb{R}^2)$ for each $p < \frac{2K}{K-1}$ and

$$\frac{1}{\pi} \int_{B_1(0)} \|\nabla u(x)\|^p \,\mathrm{d}x \leq \frac{2K}{2K - p(K-1)} \Big(-\mathbf{B}_p(z) \Big).$$

The bound is sharp: equality holds for a class of radial K-quasiregular maps.

Note: The result is due to Astala, Iwaniec, Prause & Saksman (2012) when $z = \mathrm{Id}$.

Proposition: (Guerra & JK 2021)

Let $p > 2$. Then $\mathbf{B}_p$ is quasiconvex in the following restricted sense:

If $z \in \mathbb{R}^{2\times 2}$, $\phi \in \mathrm{W}_0^{1,p}(\mathbb{X}, \mathbb{R}^2)$, then

$$\mathbf{B}_p(z + \nabla\phi) \geq 0 \quad \Rightarrow \quad \mathbf{B}_p(z) \leq \int_{\mathbb{X}} \mathbf{B}_p(z + \nabla\phi(x))\, \mathrm{d}x.$$

Notation: $\mathbb{X} \equiv \left(-\frac{1}{2}, \frac{1}{2}\right)^2$.

The limit as $p \to 2^+$: Sharp $\mathrm{L}\log\mathrm{L}$ integrability

Note: when $u \in \mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, then $\det \nabla u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.

Müller 1990: If $u \in \mathrm{W}^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, then

$$\det \nabla u \geq 0 \quad \Rightarrow \quad \det \nabla u \in \mathrm{L}\log\mathrm{L}_{\mathrm{loc}}(\Omega).$$

Generalized by Coifman, Lions, Meyer & Semmes (1992)

Corollary: (Astala, Faraco, Guerra, Koski & JK 2022)

Put

$$L(z) \equiv \|z\|^2 - \log \|z\|^2 \,\mathrm{det} z, \quad z \in \mathbb{R}^{2\times 2}.$$

L is quasiconvex in the following restricted sense:

If $z \in \mathbb{R}^{2\times 2}$ is symmetric and positive semidefinite, $\psi \in \mathrm{W}_0^{2,2}(\Omega)$, then

$$\mathrm{det}(z + \nabla^2 \psi) \geq 0 \quad \Rightarrow \quad L(z) \leq \frac{1}{\mathscr{L}^2(\Omega)} \int_\Omega L(z + \nabla^2 \psi(x)) \, \mathrm{d}x$$

Consequently, if $f \in \mathrm{W}_q^{2,2}(B_1(0))$, $q(x) = \frac{1}{2} x^T z x$ for $z \in \mathbb{R}^{2\times 2}$ symmetric and positive semidefinite and $\mathrm{det} \nabla^2 f \geq 0$ a.e. on $B_1(0)$, then

$$\frac{1}{\pi} \int_{B_1(0)} \log \|\nabla^2 f\|^2 \,\mathrm{det} \nabla^2 f \, \mathrm{d}x \leq -L(z) + \frac{1}{\pi} \int_{B_1(0)} \|\nabla^2 f\|^2 \, \mathrm{d}x.$$

The result is sharp: Equality holds for a class of radial functions.

Comments on proofs

Restricted quasiconvexity of $\mathbf{B}_p$ for $p > 2$ relies on

Extemality property: Assume $F\colon \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{+\infty\}$ satisfies

(i) $\exists\ K > 1$ such that

$$\{z:\ \|z\|^2 \leq K\det z\} \subset \mathrm{int}\big(\mathrm{dom}(F)\big)$$

(ii) F is rank-one convex and $F(\mathrm{Id}) = \mathbf{B}_p(\mathrm{Id})\ (= -1)$

(iii) F is positively p-homogeneous ($p \geq 2$)

(iv) F is isotropic.

Then $F \geq \mathbf{B}_p$ on $\{z:\ \|z\|^2 \leq K\det z\}$.

Extension of results by

- Astala, Iwaniec, Prause & Saksman (2015)
- Guerra (2018)

Comments on proofs

Restricted quasiconvexity of

$$L(z) \equiv \|z\|^2 - \log \|z\|^2 \,\mathrm{det} z, \quad z \in \mathbb{R}^{2\times 2}.$$

Locally uniformly in $z \in \mathbb{R}^{2\times 2}$:

$$\frac{p}{p-2}\Big(\mathsf{B}_p(z) - \mathsf{B}_2(z)\Big) \to L(z) \quad \text{as} \quad p \searrow 2.$$

Since $\mathsf{B}_2 = -\mathrm{det}$ the left-hand side is restricted quasiconvex on

$$\Big\{ z : \|z\|^2 \leq \tfrac{p}{p-2} \mathrm{det} z \Big\}.$$

Consequently, for each $\varepsilon > 0$ restricted quasiconvexity holds for the left-hand side when tested with $\phi(x) = \nabla \psi(x) + \varepsilon x^{\perp}$ and $2 < p < p_{\varepsilon}$.

Thanks for the attention