Linear Algebra Solutions

1. (a) [5 marks] Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$, and that $T : V \to V$ is a linear transformation.
   (i) Prove that there exists a non-zero polynomial $p(x)$ such that $p(T) = 0$.
   (ii) Prove that there exists a unique monic polynomial $m(x)$ such that for all polynomials $q(x)$, $q(T) = 0$ if and only if $m(x)$ divides $q(x)$.
   (iii) State a criterion for diagonalisability of T in terms of $m(x)$.

   (b) [10 marks] Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$ and $T : V \to V$ is a linear transformation.
   (i) Prove that for all i, $\ker T^i$ is a subspace of $\ker T^{i+1}$.
   Let $B_1 \subseteq B_2 \subseteq \cdots$ be sets such that B_i is a basis for $\ker T^i$.
   (ii) Deduce that if for some k, $T^k = 0$, then T is upper-triangularisable. Deduce that for any $\lambda \in \mathbb{F}$, if $(T - \lambda I)^k = 0$, then T is upper-triangularisable.
   (iii) Show that T is upper-triangularisable if and only if $m(x)$ is a product of linear factors.
   [*You may use the Primary Decomposition Theorem.*]

   (c) [10 marks] For which values of α and β is the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ \alpha - 1 & \alpha - \beta & \beta \\ \alpha - 1 & \alpha - \beta - 1 & \beta + 1 \end{pmatrix}$$

   diagonalisable over $\mathbb{R}$?
   For which values of α and β is it upper-triangularisable over $\mathbb{R}$?

(a) [B] (i) If the dimension of V is n, then that of $\mathrm{Hom}(V)$ is n^2. So, $\{I, T, T^2, \ldots, T^{n^2}\}$ is linearly dependent. Hence there exist constants $\alpha_0, \alpha_1, \ldots, \alpha_{n^2}$ not all zero such that

$$\sum_{i=0}^{n^2} \alpha_i T^i = 0.$$

Let $p(x) = \sum_{i=0}^{n^2} \alpha_i x^i$.

Then $p(x)$ is a non-zero polynomial and $p(T) = 0$, as required. [1 mark]

(ii) Let $p(x)$ be a non-zero polynomial of minimal degree such that $p(T) = 0$.

Define $m(x)$ to be the result of dividing $p(x)$ by its leading coefficient.

Then $m(x)$ is monic, and $m(T) = 0$. Hence if $m(x)$ divides $q(x)$, then $q(T) = 0$.

Now suppose that $q(x)$ is a polynomial such that $q(T) = 0$.

Then there exist polynomials $a(x)$ and $b(x)$ such that

$$q(x) = a(x)m(x) + b(x),$$

and $b(x) = 0$ or the degree of $b(x)$ is less than that of $m(x)$.

But then $b(x)$ must be zero, yielding that $m(x)$ divides $q(x)$, for otherwise $b(x) = q(x) - a(x)m(x)$, so $b(T) = 0$, contradicting the minimality of the degree of $m(x)$. [3 marks]

T is diagonalisable if and only if $m(x)$ is a product of distinct linear factors. [1 mark]

(b) [S; they may find (ii) and (iii) a bit harder because of the way it's presented.] (i) Suppose that $v \in \ker T^i$.

Then $T^i(v) = 0$.

Hence $T(T^i(v)) = 0$.

That is, $T^{i+1}(v) = 0$.

So $v \in \ker T^{i+1}$. [1 mark]

(ii) Writing B_k with the elements of B_1 first, followed by the elements of $B_2 \setminus B_1$, and so on, the matrix of T with respect to B_k is upper-triangular, and indeed all diagonal entries are zero.

We justify the statement that the matrix is upper-triangularisable as follows. The first few columns correspond to element of B_1, which belong to the kernel of T, and so have no non-zero entries at all. Any subsequent column corresponds to an element of $\ker T^{i+1} \setminus \ker T^i$, for some i, which is sent by T to an element of $\ker T^i$, so to a linear combination of members of the basis which are strictly earlier in the ordering. So all non-zero entries in that column are strictly above the diagonal.

(The students are very likely to draw a diagram and do their argument by reference to it.) [3 marks]

If $(T - \lambda I)^k = 0$, let $S = T - \lambda I$. Then S is upper-triangularisable. It follows immediately that T is. [1 mark]

(iii) Now suppose that

$$m(x) = \prod_{i=1}^{r} (x - \lambda_i)^{k_i}.$$

By the Primary Decomposition Theorem,

$$V = \bigoplus_{i=1}^{r} \ker(T - \lambda_i I)^{k_i}.$$

Let B_i be a basis of $\ker(T-\lambda_i I)^{k_i}$ with respect to which $T \restriction_{\ker(T-\lambda_i I)^{k_i}}$ is upper-triangularisable. Then if $B = \bigcup_{i=1}^{r} B_i$, then the matrix of T with respect to B is upper-triangular.

That $m(x)$ splits into linear factors if T is upper-triangularisable is obvious; because if $\lambda_1, \ldots, \lambda_n$ are the diagonal entries, then $\prod_{i=1}^{n}(T - \lambda_i I)$ is strictly upper-triangular (that is, all diagonal entries are zero), and therefore idempotent. Thus for some k (in fact, for some $k \leqslant n$), $\left(\prod_{i=1}^{n}(T - \lambda_i I)\right)^k = 0$. Thus $m_T(x)$ divides $\left(\prod_{i=1}^{n}(x - \lambda_i)\right)^n$, and thus splits into linear factors. [5 marks]

(c) [S/N; there's been nothing exactly like this on the paper for a few years now.] Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ \alpha - 1 & \alpha - \beta & \beta \\ \alpha - 1 & \alpha - \beta - 1 & \beta + 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \det(A - xI) &= (2 - x)((\alpha - \beta - x)(\beta + 1 - x) - \beta(\alpha - \beta - 1)) \\ &\quad - ((\alpha - 1)(\beta + 1 - x) - \beta(\alpha - 1)) \\ &\quad - ((\alpha - 1)(\alpha - \beta - 1) - (\alpha - 1)(\alpha - \beta - x)) \\ &= (2 - x)((\alpha - \beta - x)(\beta + 1 - x) - \beta(\alpha - \beta - 1)) \\ &\quad + (1 - \alpha)((\beta + 1 - x) - \beta + (\alpha - \beta - 1) - (\alpha - \beta - x)) \\ &= (2 - x)(x^2 + x(-1 - \alpha) + \alpha) \\ &= (2 - x)(x - 1)(x - \alpha). \end{aligned}$$

[1 mark]

If α is not equal to 1 or 2, then $\chi_A(x)$ has three distinct roots and so A is diagonalisable. [1 mark]

If α is equal to 1 or 2, then $\chi_A(x)$ has a repeated root, and A is diagonalisable if and only if $(A-I)(A-2I)=0$. [2 marks]

Now the $(2,1)$-entry of $(A-I)(A-2I)$ is $(\alpha-1)^2$, which is not zero unless $\alpha=1$. So if $\alpha=2$, A is not diagonalisable. [2 marks]

If $\alpha=1$, then

$$(A-I)(A-2I) = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -\beta & \beta \\ 0 & -\beta & \beta \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & -\beta-1 & \beta \\ 0 & -\beta & \beta-1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & -\beta \\ 0 & \beta & -\beta \end{pmatrix}$$

which is zero if and only if $\beta=0$. [2 marks]

So A is diagonalisable if and only if either α is not 1 or 2, or $\alpha=1$ and $\beta=0$.

By the criterion in part (b), A is upper-triangularisable whatever the values of α and β, since by the Cayley-Hamilton Theorem $m(x)$ divides $(x-2)(x-1)(x-\alpha)$ and so is a product of linear factors. [2 marks]

2. (a) [15 marks] Suppose that V is a finite-dimensional vector space over a field $\mathbb{F}$. Suppose that $B = \{e_1, \dots, e_n\}$ is a basis for V.
   (i) Define the *dual space* V' of V and the *dual basis* $B' = \{e'_1, \dots, e'_n\}$. Prove that B' is indeed a basis for V'.
   (ii) If $T : V \to V$ is a linear transformation, define the *dual map* T'. State and prove a relationship between the matrices of T and T' with respect to the bases given. How are the characteristic polynomials of T and T' related? How are the minimum polynomials related? Justify your answers briefly.
   (iii) If U is a subspace of V, define the *annihilator* U° of U.
   (iv) Define a natural isomorphism Φ between V and its double dual V''. (You do not need to give proofs that Φ is well-defined or that it is an isomorphism.) Prove that if U is a subspace of V, then $\Phi|_U$ is a bijection between U and $U^{\circ\circ}$.

   (b) [10 marks] Let V be the vector space of all functions $f : \mathbb{N} \to \mathbb{R}$ such that for all but finitely many n, $f(n) = 0$, equipped with operations of vector addition and scalar multiplication defined so that $(f + g)(n) = f(n) + g(n)$ and $(\alpha f)(n) = \alpha f(n)$ for all $f, g \in V$, $n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$.
   Define W to be the vector space of all functions from $\mathbb{N}$ to $\mathbb{R}$, with similarly defined operations of vector addition and scalar multiplication.
   If $f \in W$, define $\theta_f : V \to \mathbb{R}$ so that

$$\theta_f(g) = \sum_{n=0}^{\infty} f(n)g(n).$$

   Prove that the map $f \mapsto \theta_f$ is an isomorphism between W and V'.
   Prove that the map $\Phi : V \to V''$ defined as in part (a) is not a surjection.
   [*You may assume that if U is a vector space over $\mathbb{R}$, L is a linearly independent subset of U, and $h : L \to \mathbb{R}$, then there exists a linear functional $k : U \to \mathbb{R}$ such that $k|_L = h$.*]

(a) [B] (i) The dual space V' is the set of all linear functionals on V, that is to say, the set of all functions $f : V \to \mathbb{F}$ such that $f(u+v) = f(u) + f(v)$ and $f(\alpha v) = \alpha f(v)$ for all $\alpha \in \mathbb{F}$ and all $u, v \in V$, with vector addition and scalar multiplication defined so that $(f+g)(v) = f(v)+g(v)$ and $(\alpha f)(v) = \alpha f(v)$ for all $v \in V$, $f, g \in V'$ and $\alpha \in \mathbb{F}$. [1 mark]

The dual basis is defined so that $e'_i(e_j) = \delta_{i,j}$. [1 mark]

The dual basis is linearly independent, since if

$$\alpha_1 e'_1 + \cdots + \alpha_n e'_n = 0,$$

then for all i,

$$(\alpha_1 e'_1 + \cdots + \alpha_n e'_n)(e_i) = 0,$$

that is, $\alpha_i = 0$.

To prove that it is a spanning set, suppose that $f \in V'$. Let $\alpha_i = f(e_i)$ for all i. Then for all i,

$$f(e_i) = \alpha_i = \Big(\sum_j \alpha_j e'_j\Big) e_i$$

so since f and $\sum_j \alpha_j e'_j$ are linear and agree on a spanning set, they are equal. [3 marks]

(ii) If $f \in V'$, then define $T'(f)$ so that $T'(f)(v) = f(T(v))$ for all $v \in V$. [1 mark]

Let the matrix of T with respect to B be $(a_{i,j})$ and the matrix of T' with respect to B' be $(b_{i,j})$.

Then

$$e_i'(T(e_j)) = e_i'\Big(\sum_{k=1}^{n} a_{k,j}e_k\Big) = a_{i,j},$$

while

$$(T'(e_i'))(e_j) = \Big(\sum_{k=1}^{n} b_{k,i}e_k'\Big)(e_j) = b_{j,i}.$$

So $b_{j,i} = a_{i,j}$, and the matrices are each other's transpose; and so their minimum polynomials are the same, as are their characteristic polynomials. [4 marks]

(iii) $U^\circ = \{f \in V' : \forall u \in U\, f(u) = 0\}$. [1 mark]

(iv) Φ is defined so that for all $f \in V'$ and $v \in V$,

$$\Phi(v)(f) = f(v).$$

We show that $u \in U$ if and only if for all $f \in U^\circ$, $f(u) = 0$.

The forward direction is simply the definition of U°.

As for the reverse direction, let $\{e_1, \ldots, e_k\}$ be a basis for U and extend it to a basis $\{e_1, \ldots, e_n\}$ for V. Let $\{e_1', \ldots, e_n'\}$ be the dual basis. Then $(\sum_{j=1}^{n} \alpha_j e_j')(e_i) = 0$ if and only if $\alpha_i = 0$. It follows that $f(e_i) = 0$ for all $i < n$ if and only if f is in the span of $\{e_{k+1}', \ldots, e_n'\}$. It now readily follows that U° is the span of $\{e_{k+1}', \ldots, e_n'\}$.

Now, $u \in U$ if and only if for all $f \in U^\circ$, $f(u) = 0$, if and only if for all $f \in U^\circ$, $\Phi(u)(f) = 0$, if and only if $\Phi(u) \in U^{\circ\circ}$. [4 marks]

(b) [N] If $f \in W$, we observe that θ_f is linear, so is an element of V'. Also, if $f \neq 0$, then there exists $n \in \mathbb{N}$ such that $f(n) \neq 0$. Now we define $g \in V$ such that $g(n) = 1$, and $g(m) = 0$ for all $m \neq n$. Then $\theta_f(g) = f(n) \neq 0$. So the operator $f \mapsto \theta_f$ is one-to-one. Finally, to show that it is onto, let h be any element of V'. Then if g_n is defined, for each natural number n, so that $g_n(m) = 1$ if $m = n$ and is equal to 0 otherwise, then the set of g_n is a basis for V. So if f is defined so that $f(n) = h(g_n)$ for each n, then for any $g \in V$, $g = \sum_n g(n)g_n$, and $\theta_f(g) = \sum_n f(n)g(n) = \sum_n h(g_n)g(n) = h(\sum_n g(n)g_n) = h(g)$. So $h = \theta_f$. [4 marks]

For each n, define $f_n(m)$ to be 1 if $n = m$ and 0 if $n \neq m$. Let g be the function $n \mapsto 1$. Then $\{f_n : n \in \mathbb{N}\} \cup \{g\}$ is linearly independent in W, and so its image under the operator $f \mapsto \theta_f$ is linearly independent in V'.

Define $h(\theta_{f_n})$ to be 0 and $h(g)$ to be 1. Extend this to a linear functional k on V'.

Since $k(\theta_{f_n}) = 0$ for all n and $k(g) = 1$, k cannot be in the image of Φ.

[6 marks]

3. Let V be a finite-dimensional inner-product space over $\mathbb{C}$.
   (a) [6 marks] Suppose that $T : V \to V$ is a linear transformation. Define the *adjoint* map T^*.
   Suppose that T has the property that $T^* = \alpha T$ for some $\alpha \in \mathbb{C}$. Prove that T is diagonalisable.
   (b) [9 marks] We say that T is *self-adjoint* if $T^* = T$, and that it is *skew-adjoint* if $T^* = -T$. Observe that if S and T are self-adjoint, then so are $S + T$, $S - T$, and βT, for any real number β.
   Recall that if $T : V \to V$ is any linear transformation, then $T + T^*$ is self-adjoint.
      (i) Prove that any linear transformation T can be written as the sum of a self-adjoint and a skew-adjoint linear transformation.
      Is it the case that a sum of diagonalisable linear transformations is diagonalisable? Give a proof or a counterexample.
      (ii) What are the possible eigenvalues of a self-adjoint linear transformation? Justify your answer carefully.
      (iii) Characterise the possible Jordan Normal Forms of linear transformations $T : V \to V$ such that T^2 is self-adjoint.
   (c) [10 marks] Suppose now that $T : V \to V$ is a linear transformation, and that $TT^* = T^*T$.
      (i) Prove that if v is an eigenvector of T^*, then $v^\perp$ is T-invariant.
      (ii) Prove that if $V_\lambda = \ker(T - \lambda I)$, and $v \in V_\lambda$, then $T^*v \in V_\lambda$ also.
      (iii) Hence prove that there exists an orthogonal basis for V consisting of vectors which are eigenvectors for both T and T^*.
      (iv) Does it follow that T is self-adjoint? Give a proof or a counterexample.

(a) [B/S] The *adjoint* is the unique linear transformation $T^* : V \to V$ such that for all $u, v \in V$, $(T^*v, u) = (v, Tu)$. [1 mark]

Suppose that $T^* = \alpha T$, where $\alpha \neq 0$. Assume that V is not trivial. Since the underlying field is $\mathbb{C}$, $\chi_T(x)$ has a root, so T has an eigenvector, v; say λ is the eigenvalue.

We prove that $v^\perp$ is T-invariant.

Suppose that $u \in v^\perp$.

Then $(u, v) = 0$.

Also $(Tu, v) = (\lambda u, v) = \lambda(u, v) = 0$.

So $(u, T^*v) = 0$.

Now $T^*v = \alpha Tv$, so $(u, \alpha Tv) = 0$, so $\overline{\alpha}(u, Tv) = 0$, so since $\alpha \neq 0$, $(u, Tv) = 0$ as required.

By the inductive hypothesis we assume that $T \restriction_{v^\perp}$ has a basis B of eigenvectors. Then $B \cup \{v\}$ is a basis of eigenvectors for T. [5 marks]

(b) [B/N]

(i) $T - T^*$ is clearly skew-self-adjoint.

$T = (1/2)(T + T^*) + (1/2)(T - T^*)$ as required. [1 mark]

The linear transformation with matrix with respect to the standard basis given by

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalisable, since its characteristic polynomial is x^2 and its minimum polynomial is not x.

But it is the sum of a self-adjoint and a skew-self-adjoint transformation as above. [3 marks]

(ii) I is certainly self-adjoint so for all real β, βI is self-adjoint also, and has eigenvalue β.

Conversely, if T is self-adjoint with eigenvalue λ, then $(Tv, v) = (\lambda v, v) = \lambda\|v\|^2$, while $(v, Tv) = (v, \lambda v) = \overline{\lambda}\|v\|^2$, so $\lambda = \overline{\lambda}$ and λ is real. [1 mark]

(iii) [N] Suppose that T^2 is self-adjoint and

$$A = \begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & & \\ & & & \\ & & & \lambda \end{pmatrix}$$

is a Jordan block for T.

Then A^2 has the form

$$A = \begin{pmatrix} \lambda^2 & 2\lambda & 1 & \dots & 0 \\ 0 & \lambda^2 & 2\lambda & & \\ & & & & \\ & & & & \lambda^2 \end{pmatrix}$$

and is diagonal if and only if either the size of the block is 1×1, or it has size 2×2 and $\lambda = 0$.

Also, A^2 is diagonalisable if and only if it is diagonal; for if it is not diagonal then its minimum polynomial is $(x - \lambda^2)^k$ for some $k > 1$, which is not a product of distinct linear factors. [3 marks]

So the Jordan Normal Forms of transformations T such that T^2 is self-adjoint have Jordan blocks of that form, with λ being either real or purely imaginary. [1 marks]

(c) (i) Suppose v is an eigenvector of T^*, and $u \in v^\perp$.

Then $(v, u) = 0$.

Since T^*v is a scalar multiple of v, $(T^*v, u) = 0$.

Hence $(v, Tu) = 0$, and so $Tu \in v^\perp$, as required. [2 marks]

(ii) Suppose that $v \in V_\lambda$.

Then $T^*Tv = T^*(\lambda v) = \lambda T^*v$. But also $T^*Tv = TT^*v$. Hence $T(T^*v) = \lambda T^*v$, and so $T^*v \in V_\lambda$. [2 marks]

(iii) If V is non-trivial, then the characteristic polynomial of T, being a non-constant complex polynomial, has a root. So T has an eigenvalue λ, whose corresponding eigenspace V_λ is non-trivial. Now $T^*|_{V_\lambda}$ also has an eigenvector by the same reasoning, which is a simultaneous eigenvector of T and T^*. [2 marks]

We do induction on $\dim V$.

Let u be a simultaneous eigenvector for T and T^*. Then $u^\perp$ is invariant under both T^* and T.

By the inductive hypothesis, $u^\perp$ has a basis B of the correct form.

Then $B \cup \{u\}$ is a basis of the desired form for V. [2 marks]

(iv) If $T = \mathrm{i}I$, then $T^* = -\mathrm{i}I$. These commute, but are not equal. [2 marks]