A0 Linear Algebra Solutions

[B] Bookwork, [S] Seen similar, [N] New

Question 1 (a) (i) The minimal polynomial $m_T(x)$ is defined to be the monic polynomial $f(x)$ of least degree such that $f(T) = 0$. It exists since the Cayley-Hamilton theorem states that $\chi(T) = 0$ where $\chi(x) = \det(xI - T)$ is the characteristic polynomial of T. [1 mark B]

Now if λ is an eigenvalue of T with eigenvector v, then $0 = m_T(T)(v) = m_T(\lambda)v$ and hence $m_T(\lambda) = 0$. Conversely if λ is a root of m_T then $m_T = (x - \lambda)g(x)$ with $\deg g(x) < \deg m_T$. Therefore $g(T) \neq 0$ and we can find a nonzero vector v such that $w := g(T)v \neq 0$. But then $(T - \lambda)w = (T - \lambda)g(T)v = m_T(T)v = 0$ and so λ is an eigenvalue of T with eigenvector w. [3 marks, B]

(ii) If T is diagonalizable with respect to some basis B so that $X = {}_B[T]_B$ is a diagonal matrix, we consider a polynomial $f(x)$ to be product of linear factors $(x - \mu)$ where μ ranges over the distinct diagonal entries of X. We see $f(T) = 0$ and together with (a)(i) we deduce that $f = m_T$ and has distinct roots. Conversely suppose $m_T = \prod_{i=1}^{k}(x - \mu_k)$ has distinct roots. We argue by induction on k, the case $k = 1$ being clear. Let $m_T(x) = (x - \mu_1)g(x)$. Then $g(\mu_1) \neq 0$. Let $U_1 = \ker(T - \mu_1)$, $U_2 = \ker g(x)$. If $v \in U_1 \cap U_2$ then $0 = g(T)v = g(\mu_1)v$ and so $v = 0$ since μ_1 is not a root of $g(x)$. We now show $V = U_1 + U_2$. Let $v \in V$ and define $v_1 = g(\mu_1)^{-1}g(T)v, v_2 = v - v_1$. Since $m_T(v) = 0 = (T - \mu_1)g(T)v$ it follows that $Tv_1 = \mu_1 v$, i.e. $v_1 \in U_1$. Also $g(T)v_2 = g(T)v - g(T)v_1 = g(T)v - g(\mu_1)v_1 = 0$ by the definition of v_1. So $v_2 \in U_2$. Therefore $V = U_1 \oplus U_2$. Now T acts as the scalar μ_1 on U_1 and $g(T) = 0$ on U_2 hence the minimal polynomial of T on U_2 has degree less than k. By induction we can choose a basis diagonalizing T on U_2 and adding any basis of U_1 we are finished. [8 marks, B]

(iii) Let v be an eigenvector of T, and note that v is still an eigenvector for any $p(T)$. We can extend v to a basis $v, w_1, w_2 \ldots$ of V and define $B(v) = w_1$ and $B(w_i) = 0$. Then v is not an eigenvector of B and so $B \neq p(T)$ for any polynomial $p(x)$. The students can also argue using that the dimension of the space spanned by $\{1, T, T^2, \ldots, \}$ in $End(V)$ is exactly $\deg m_T \leq n$. [3 marks, N]

(b)(i) If A is diagonalizable, then for some change of basis matrix P the matrix $P^{-1}AP$ is a diagonal matrix with eigenvalues λ, λ^{-1} for some nonzero $\lambda \in \mathbb{C}$. Take $\mu \in \mathbb{C}$ such that $\mu^2 = \lambda$ and let X be the diagonal matrix with entries μ, μ^{-1}. Take $B = P^{-1}XP$. [3 marks, S]

(ii) Take $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. Suppose $A = B^2$. The eigenvalues of B must be $\pm i$ and since $\det(B) = 1$ they must be distinct: i and $-i$. By part (a) B is similar to a diagonal matrix with diagonal entries $\{i, -i\}$ and so $B^2 = -Id \neq A$. Contradiction. [3 marks, N]

(iii) $SL(2, \mathbb{F})$ is a finite set so if the map $A \mapsto A^2$ is surjective it must be injective. But this is not true, as for odd p we have $Id^2 = (-Id)^2$ while if $p = 2$

then

$$Id^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 .$$

[4 marks, N]

Question 2

(a) (i) Suppose $B = \{b_1, \dots, b_n\}$ is a basis of V and $C = \{c_1, \dots, c_m\}$ is a basis of W. We define $B' = \{b'_1, \dots, b'_n\}$ where $b'_i \in V'$ is such that $b'_i(b_j) = \delta_{ij}$. [1 mark, B]

Similarly we take $C' = \{c'_1, \dots, c'_m\}$. Suppose ${}_C[T]_B = (a_{ij})$ so that $T(b_j) = \sum_{i=1}^m a_{ij} c_i$. We compute $T'(c'_j)(b_s) = c'_j(T(b_s)) = c'_j(\sum_{i=1}^m a_{is} c_i) = a_{js}$ This gives $T'(c'_j) = \sum_{i=1}^n a_{ji} b'_i$ and so ${}_{B'}[T']_{C'}$ is the transpose of ${}_C[T]_B$. [4 marks, B]

(ii) For a polynomial $f(x)$ and a square matrix X we have $f(X^t) = (f(X))^t$ and so $f(T) = 0$ if and only if $f(X^t) = 0$. It follows that $m_T = m_{T'}$. Let $A =_B [T]_B$. Then $\chi_T(x) = \det(xId - A) = \det(xId - A)^t = \det(xId - A^t) = \chi_{T'}(x)$. [2 marks, S]

(iii)We define $U^0 := \{f \in V' \mid f(u) = 0, \forall u \in U\}$.

Let $b_1, \dots, b_k$ be a basis of U and extend this to a basis $B = \{b_1, \dots, b_n\}$ of V. We claim that U^0 has basis $b'_{k+1}, \dots, b'_n$. Indeed these functionals are linearly independent (since they are a subset of B'), and for $f = \sum_{i=1}^n \alpha_i b'_i$ the condition $f \in U^0$ is equivalent to $f(b_i) = 0$ for $i = 1, \dots, k$, which is equivalent to $\alpha_1 = \dots = \alpha_k = 0$. This proves the claim. Hence $\dim U^0 = n - k = \dim V - \dim U$ and we are done. [4 marks, B]

(b) (i) There are many ways to argue this, here is an argument which also applies to (ii).

We note that $(Im(T))^0 = \ker T'$. Indeed $f \in (Im(T))^0$ iff $f(Tv) = 0$ for all $v \in V$ iff $f \circ T = 0$ iff $f \in \ker T'$. Now by part (a) (iii) and the Rank-Nullity theorem applied to T' we have

$$\dim Im(T) = \dim W - \dim(Im(T))^0 = \dim W' - \dim \ker T' = \dim Im(T').$$

[4 marks, S]

(ii) The above argument only uses that $\dim W$ is finite, so the result remains true even if $\dim V$ is infinite. [3 marks, N]

(c) Let $U = \cap_{i=1}^k \ker f_i$ and let L be the subspace of V' spanned by all f_i. Observe that $U = \cap_{h \in L} \ker h$. Now if $L = V'$ then choosing a basis B of V we consider the dual basis B' and then $U = \cap_{b \in B'} \ker b = \{0\}$. [2 marks, S]

For the converse the students may argue using the natural isomorphism between V and V''. Here is an alternative short argument: Suppose $\dim V = n$ and $L \neq V'$. Choose a basis $g_1, \dots, g_k$ of L and note $k < n$. Thus

$$U = \cap_{i=1}^k \ker g_i = \ker \phi,$$

where $\phi : V \to \mathbb{F}^k$ is the linear map $\phi(v) = (g_1(v), g_2(v), \dots, g_k(v))$. Since $k < \dim V$ the Rank-Nullity Theorem applied to ϕ gives that $U = \ker \phi \neq \{0\}$. Contradiction, therefore $L = V'$. [5 marks, N]

Question 3

(a) If $v \in U \cap U^\perp$ then $\langle v, v\rangle = 0$ and hence $v = 0$ since the inner product is positive definite. Hence $U \cap U^\perp = \{0\}$. We now show that $V = U + U^\perp$ Let $v \in V$. Let $e_1, \dots, e_k$ be an orthonormal basis of U and define $v_1 = \sum_{i=1}^k \langle v, e_i\rangle e_i$. Then $v_1 \in U$ and $\langle v, e_i\rangle = \langle v_1, e_i\rangle$ for all i which implies that $v - v_1$ is orthogonal to each e_i, i.e. $v - v_1 \in U^\perp$. Hence $V = U + U^\perp$ and therefore $V = U \oplus U^\perp$. [5 marks, B]

(b) (i) The adjoint N^* is the unique linear transformation $N^* : V \to V$ such that $\langle N^*(v), w\rangle = \langle v, N(w)\rangle$ for all $v, w \in V$. [1 Mark, B]

(ii) Fix $w \in U^\perp$ and let $v \in U$. We have $\langle N^*(w), v\rangle = \langle w, N(v)\rangle = 0$ since $N(v) \in U$. This holds for all $v \in U$ and hence $N^*(w) \in U^\perp$. The vector $w \in U^\perp$ was arbitrary and so $N^*(U^\perp) \subseteq U^\perp$. [4 marks, S]

(iii) If $N = S + A$ as required then $N^* = S - A$ and so we can solve $S = (N + N^*)/2, A = (N - N^*)/2$. This A and S are uniquely determined by N. Conversely we check $(\frac{N+N^*}{2})^* = \frac{N+N^*}{2}$ and $(\frac{N-N^*}{2})^* = -\frac{N-N^*}{2}$ so A and S exist for any N.

Now if $NN^* = N^*N$ then we check

$$\frac{N + N^*}{2}\frac{N - N^*}{2} = \frac{N^2 - (N^*)^2}{2} = \frac{N - N^*}{2}\frac{N + N^*}{2}$$

Conversely if A and S commute than $NN^* = (A + S)(S - A) = S^2 - A^2 = (S - A)(A + S) = N^*N$ [5 marks, S]

(c) (i) Suppose $v \in \ker N$. Then $||N^*(v)||^2 = \langle N^*v, N^*v\rangle = \langle v, NN^*v\rangle = \langle v, N^*Nv\rangle = 0$ and so $N^*(v) = 0$ giving that $v \in \ker N^*$. Hence $\ker N \subseteq \ker N^*$ The same argument applied with N^* instead of N gives the opposite containment and hence $\ker N^* = \ker N$. [3 marks, N]

Let $U = \ker N = \ker N^*$. Since both N and N^* send U into U, from part (b)(ii) we have $N(U^\perp) \subseteq U^\perp$ and $N^*(U^\perp) \subseteq U^\perp$. Also both N and N^* are injective when restricted to $U^\perp$ and hence both maps are bijections when restricted to $U^\perp$. Finally $N(V) = N(U + U^\perp) = N(U^\perp) = U^\perp$ and arguing with N^* in place of N we get $Im(N) = Im(N^*) = U^\perp$. [3 marks, N]

(ii) We have $||N^*(v)|| = \langle v, NN^*(v)\rangle$ and $||N(v)|| = \langle v, N^*N(v)\rangle$. Therefore if we set $A = NN^* - N^*N$ we get $\langle v, Av\rangle = 0$ for all $v \in V$ From this point the students can argue with the spectral theorem to deduce $A = 0$ but there is a direct way: Let $u, v \in V$ and apply the above equality to $u + v$. So $0 = \langle u + v, A(u + v)\rangle$. Using $\langle v, Av\rangle = \langle u, A(u)\rangle = 0$ we obtain

$$\langle u, A(v)\rangle + \langle v, A(u)\rangle = 0$$

Now replace v with iv to obtain

$$i\langle u, A(v)\rangle - i\langle v, A(u)\rangle = 0$$

Solving the two equations we get $\langle u, A(v)\rangle = 0$ for all $u, v \in V$ and so $A = 0$ and $NN^* = N^*N$. [4 marks, N]