DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A1 Mathematical Methods I

HILARY TERM 2018

THURSDAY, 11 JANUARY 2018, 9.30am to 11.30am

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question in a new booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Applied Partial Differential Equations

1. (a) [10 marks] Consider the PDE

$$au_x + bu_y = c \tag{1}$$

where a, b, and c are constants.

(i) By applying the method of characteristics with arbitrary initial data, give a parametric form of the general solution.

(ii) Obtain the general solution as the intersection of two families of surfaces by integrating

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

(iii) Show that the two forms of solution obtained in parts (i) and (ii) are equivalent.

(b) [15 marks] Consider the following first order PDE

$$xu_x + yu_y = 1,$$

along with boundary data $u = 0$ on the curve $y = x^2 + 1$, $x > 0$.

(i) Identify two separate segments of the boundary curve for which the data is Cauchy.

(ii) Obtain an explicit solution $u(x, y)$ for each segment, and give the domain of definition in each case.

2. (a) [10 marks] Given an nth order hyperbolic PDE for $\mathbf{u}(t,x) \in \mathbb{R}^n$

$$\frac{\partial}{\partial t}\mathbf{P}(t,x,\mathbf{u}) + \frac{\partial}{\partial x}\mathbf{Q}(t,x,\mathbf{u}) = \mathbf{0},$$

where $\mathbf{P}$ and $\mathbf{Q}$ are vector-valued functions, state the Rankine-Hugoniot condition for the slope of a shock and determine the number of incoming characteristics for the shock to be causal.

(b) [15 marks] Consider the following first order PDE

$$(t+1)u\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0, \quad t > 0 \tag{2}$$

along with data

$$u(x,0) = \begin{cases} 1 & \text{for } x < 1, \\ 3 & \text{for } x > 1. \end{cases}$$

(i) Show that (2) can be written in conservation form

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = R$$

with

$$P = \frac{1}{2}(t+1)u^2, \quad Q = xu.$$

Obtain the form of R.

(ii) Show that a shock exists along a curve $t = C(x)$, which you should determine.

(iii) Sketch the characteristic projections and shock curve. Is the shock causal?

3. (a) [12 marks] Consider the first order PDE system

$$\mathbf{A}(x, y, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x} + \mathbf{B}(x, y, \mathbf{u})\frac{\partial \mathbf{u}}{\partial y} = \mathbf{c},$$

where $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{A}, \mathbf{B}$ are given n by n matrices with smooth components.

(i) Defining characteristics as curves $\lambda = \frac{dy}{dx}$ across which $\mathbf{u}$ is continuous but there can be jumps in $\mathbf{u}_x$ and $\mathbf{u}_y$, derive the condition

$$\det(\mathbf{B} - \lambda\mathbf{A}) = 0. \tag{3}$$

(ii) What does it mean for the system to be hyperbolic?

(iii) Let $n = 2$ and suppose characteristics satisfy $\lambda^+ = 1$, $\lambda^- = -2$. Suppose $\mathbf{u}$ is given on the data curve $x = 0$, $0 \leqslant y \leqslant 2$, and that the solution remains bounded. Sketch the domain of definition.

(b) [13 marks] Consider the system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = (u - v)^2.$$

(i) Show that the characteristic directions are given by $\lambda = \frac{dy}{dx} = \pm 1$, and obtain the ODEs satisfied along the characteristics.

(ii) Obtain an explicit general solution for u and v in terms of two arbitrary functions.

4. Suppose that $u(x, y)$ satisfies

$$\mathcal{L}u := \frac{\partial^2 u}{\partial x \partial y} + a\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) + bu = 0, \tag{4}$$

where a and b are known functions of x and y, and

$$x = x_0(s),\ y = y_0(s),\ u = u_0(s),\ \frac{\partial u}{\partial n} = v_0(s), \tag{5}$$

on a smooth and differentiable curve $\Gamma(s) = \Gamma(x_0(s), y_0(s))$.

(a) [10 marks] Formulate a problem for the Riemann function $R(x, y; \xi, \eta)$ and determine an integral representation for the solution of (4), (5) in terms of the Riemann function R, u and its partial derivatives on $\Gamma(s)$, and the function a.

[*You may use without proof the identity*

$$\begin{aligned} R\left[u_{xy} + au_x + au_y + bu\right] - u\left[R_{xy} - \partial_x(aR) - \partial_y(aR) + bR\right] \\ = \partial_x\left(Ru_y + auR\right) + \partial_y\left(-uR_x + auR\right).\end{aligned}$$]

(b) [15 marks] Consider the following partial differential equation for $U(z, t)$:

$$z^2\left(\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial z^2}\right) - 4z\,\frac{\partial U}{\partial z} - 2U = 0$$

(i) Restricting to the domain $z > 0$, determine the characteristic coordinates $x(z, t)$, $y(z, t)$, and transform the problem to canonical form, so that $U(z, t) = u(x, y)$, say.

(ii) Determine the Riemann function $R(x, y; \xi, \eta)$ for the transformed problem.
[*Hint: seek a solution in the form*

$$R(x, y; \xi, \eta) = f\left(\frac{x + y}{\xi + \eta}\right).$$]

Section B: Supplementary Mathematical Methods

5. (a) The differential operator L is defined by ($' = \mathrm{d}/\mathrm{d}x$)

$$Ly \equiv y''(x)$$

on $0 < x < 1$.

(i) [8 marks] Find the eigenvalues λ_k and corresponding eigenfunctions y_k of

$$Ly_k = \lambda_k y_k$$

with boundary conditions

$$y(0) = 0, \quad y'(1) = 0.$$

(ii) [7 marks] Determine for which α and β the boundary value problem

$$y'' + \frac{\pi^2}{4} y = 0, \qquad y(0) = \alpha, \qquad y'(1) = \beta,$$

has solutions, and when the solution is unique.

(b) The differential operator M is defined on $-1 < x < 1$ by

$$My \equiv \begin{cases} y''(x) & \text{for } -1 < x < 0 \\ y''(x) - y(x) & \text{for } 0 \leqslant x < 1 \end{cases}$$

with boundary conditions

$$y(-1) = 0, \quad y(1) = 0.$$

(i) [5 marks] Show that the bounded eigenfunctions y for

$$My = \lambda y$$

satisfy

$$[y']_-^+ = 0, \qquad [y]_-^+ = 0 \quad \text{at } x = 0,$$

where $[y]_-^+ = \lim_{\varepsilon \to 0} y(\varepsilon) - \lim_{\varepsilon \to 0} y(-\varepsilon)$, $\varepsilon > 0$, denotes the jump of y across $x = 0$.

(ii) [5 marks] Give approximate values for large negative eigenvalues λ.
[Hint: Formulate an approximate eigenvalue problem for large negative eigenvalues λ, giving reasons for your choice.]

6. (a) (i) [5 marks] Find the general solution of the linear differential equation:

$$Ly \equiv 4\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 0, \tag{6}$$

for $0 < x < 2$.

(ii) [11 marks] Consider the boundary value problem

$$Ly(x) = f(x), \quad 0 < x < 2, \qquad y(0) + 2\frac{\mathrm{d}y}{\mathrm{d}x}(0) = 0, \quad 3y(2) - 2\frac{\mathrm{d}y}{\mathrm{d}x}(2) = 0, \tag{7}$$

with Ly as in (6). Write down two equivalent problems for the Green's function $g(x, \xi)$:
(I) using the delta function $\delta(x)$;
(II) using only classical functions and with appropriate conditions at $x = \xi$.
Determine $g(x, \xi)$ explicitly.

(b) [9 marks] State what it means that a sequence of distributions u_N, $N = 1, 2, \ldots$ converges to another distribution u as $N \to \infty$.
For integer $N \geqslant 0$, let

$$f_N(x) = \begin{cases} \displaystyle\sum_{j=1}^{N} (-1)^j \sin((2j+1)x) & \text{for } 0 < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f_N converges to $\alpha\delta(x - \pi/2)$, where δ is the delta distribution, and α is a constant that you need to determine.
[Hint: You can expand a test function ϕ into a sine series $\phi(x) = \sum_{j=1}^{\infty} c_j \sin(jx)$, and you can use, without proof, that this series converges pointwise for every $0 < x < \pi$.]