DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A1 Mathematical Methods I

HILARY TERM 2019

THURSDAY, 10 JANUARY 2019, 9.30am to 12.00pm

You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best answer(s), making a total of four answers.

Please start the answer to each question in a new booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Applied Partial Differential Equations

1. (a) [13 marks] Consider the semilinear first order PDE

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u).$$

Suppose data is prescribed on the curve Γ, that is $u = u_0(s)$ on

$$x = x_0(s), \quad y = y_0(s),$$

where s parameterises Γ.

(i) Show that u_x and u_y can be uniquely determined along Γ if

$$ay_0'(s) - bx_0'(s) \neq 0. \tag{1}$$

(ii) Show that (1) is equivalent to Γ not being tangent to a characteristic projection.

(iii) Show that the second derivatives u_{xx}, u_{xy}, and u_{yy} can also be determined uniquely if condition (1) is met.

(iv) Let $a(x, y) = x$ and $b(x, y) = 1$. Show that for any data satisfying (1), the solution will not not become multivalued regardless of the function c.

(b) [12 marks] Consider the PDE

$$u_x - 4uxu_y = 2x,$$

with boundary data $u = y^2/4$ on $x = 0$.

Obtain a parametric solution, and provide a sketch of the domain of definition.

2. Consider a PDE of the form

$$u + F(p, q) = 0, \tag{2}$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$.

(a) [7 marks] (i) In Charpit's equations, characteristic projections are defined by

$$\dot{x} = F_p, \ \dot{y} = F_q,$$

where overdot denotes derivative with respect to the characteristic parameter τ. Derive equations for $\dot{u}$, $\dot{p}$, and $\dot{q}$.

(ii) Give explicit forms for p and q.

(b) [8 marks] Consider the partial differential equation

$$2u_x u_y = u$$

with data

$$u = \beta \text{ on } x = x_0(s), \ y = y_0(s),$$

where β is a constant.

(i) Derive a necessary condition on β, x_0, and y_0 for there to exist a real solution.

(ii) Find all solutions satisfying the data $u = -1$ on $y = x$. You may leave your solutions in parametric form.

(c) [10 marks] Let $F = \frac{-(p^2+q^2)}{2}$ in (2). Let the boundary curve Γ be given by the ellipse

$$x_0(s) = a \cos s, \ y_0(s) = b \sin s,$$

and suppose that the characteristic projections intersect Γ orthogonally. Given that at the point $s = \pi/4$, $u_0 = 1$, give an explicit parameterisation of the characteristic projection $(x(\pi/4, \tau), y(\pi/4, \tau))$ that leaves this point and moves outside the ellipse.

3. (a) [12 marks] Consider the partial differential equation

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0,$$

in the region $x > y$. Reduce this equation to canonical form and thus find the general solution.

(b) [13 marks] Let $\mathbf{A}(x, y)$ and $\mathbf{B}(x, y)$ be 2×2 matrices and let $\mathbf{u} = (u(x, y), v(x, y))$ satisfy the partial differential equation system

$$\mathbf{A}\mathbf{u}_x + \mathbf{B}\mathbf{u}_y = \mathbf{c}$$

for $\mathbf{c}(x, y)$ a smooth vector-valued function.

(i) State a condition on $\mathbf{A}$ and $\mathbf{B}$ for which the system is parabolic.

(ii) Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \alpha & 1 \\ 1 & \beta \end{pmatrix} \tag{3}$$

Show that the system is hyperbolic for all α, β.

(iii) Now let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & \beta \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} \gamma v \\ u \end{pmatrix}, \tag{4}$$

where $\beta > 0$ and γ are constants. Obtain a relationship between β and γ for which the system may be solved explicitly and in this case give the explicit general solution in terms of two arbitrary functions of x and y.

4. (a) [7 marks] Let $u(x, y)$ satisfy

$$\begin{aligned} u_{xx} + u_{yy} &= f(x, y) \quad \text{in } \Omega \\ u + \frac{\partial u}{\partial n} &= g(x, y) \quad \text{on } \partial\Omega \end{aligned}$$

where Ω is a closed simply connected domain with smooth boundary $\partial\Omega$. Show that if a solution exists, it is unique.

(b) [8 marks] Consider now the Neumann problem for Laplace's equation

$$\begin{aligned} &\nabla^2 u \equiv u_{xx} + u_{yy} = f(x, y) \ \text{ in } \Omega \\ &\frac{\partial u}{\partial n} = g(x, y) \ \text{ on } \partial\Omega \end{aligned} \tag{5}$$

where again f and g are given smooth functions and Ω is a closed region. Determine a value of the constant c for which

$$\begin{aligned} &\nabla^2 G = \delta(x - \xi)\delta(y - \eta) + c \ \text{ in } \Omega \\ &\frac{\partial G}{\partial n} = 0 \ \text{ on } \partial\Omega \end{aligned}$$

provides a well-posed problem for the Green's function $G(x, y; \xi, \eta)$.

(c) [10 marks] Construct via the Method of Images and infinite series the Green's function $G(\mathbf{x}; \xi)$ satisfying

$$\begin{aligned} &\nabla^2 G = \delta(\mathbf{x} - \xi) \ \text{ in } 0 < y < h, \ x \in \mathbb{R} \\ &G = 0 \text{ on } y = 0, \\ &\frac{\partial G}{\partial y} = 0 \text{ on } y = h. \end{aligned}$$

[*You do not need to consider convergence of the series.*]

Section B: Supplementary Mathematical Methods

5. (a) [5 marks] For a differential operator L with homogeneous boundary conditions, give the definition of its *full adjoint* L^*, explaining your notation carefully.

   (b) [8 marks] Consider the Sturm-Liouville operator

$$Ly(x) \equiv -(p(x)y(x)')' + q(x)y(x), \qquad a < x < b, \tag{6}$$

   where the prime denotes derivatives with respect to x, with boundary conditions

$$y(a) = 0, \qquad y'(b) = \alpha y'(a).$$

   Determine the adjoint operator L^* including the adjoint boundary conditions and from this derive sufficient and necessary conditions on α for L to be fully self-adjoint.

   (c) [12 marks] Convert the boundary value problem

$$-y''(x) + 2y'(x) - y(x) = f(x), \qquad y(0) = 0, \quad y'(1) = 0,$$

   into Sturm-Liouville form $Ly = rf$ with L as in (6) and a weighting function r.
   Let λ be one of the eigenvalues of L with corresponding eigenfunction y, i.e. λ and $y(x) \not\equiv 0$ satisfy

$$Ly(x) = \lambda r(x)y(x).$$

   Derive an algebraic equation for λ in the form $g(\sqrt{\lambda}) = -\sqrt{\lambda}$, where you have to determine the function g. Show, by drawing a sketch or otherwise, that there is exactly one eigenvalue $\lambda = \lambda_n$ in the interval

$$\left[\left(-\frac{\pi}{2} + n\pi\right)^2, \left(\frac{\pi}{2} + n\pi\right)^2\right] \qquad \text{for } n = 1,\ 2,\ 3, \ldots,$$

   and that there are no eigenvalues $\lambda \leqslant 0$. What is the limit for $\sqrt{\lambda_n} - n\pi$ as $n \to \infty$?

6. (a) (i) [6 marks] For the boundary value problem

$$Ly \equiv y''(x) - 3y'(x) + 2y(x) = f(x) \quad \text{for } 0 < x < 1, \quad y'(0) = \alpha, \quad y'(1) - y(1) = \beta, \tag{7}$$

where $f(x)$ and the constants α and β are given, and primes denote derivatives with respect to x, define the Green's function $g(x, \xi)$ using the delta function $\delta(x)$, and verify that

$$g(x,\xi) = \begin{cases} \left(2\mathrm{e}^{x} - \mathrm{e}^{2x}\right)\mathrm{e}^{-2\xi} & \text{for } 0 < x \leqslant \xi < 1, \\ \left(-\mathrm{e}^{-\xi} + 2\mathrm{e}^{-2\xi}\right)\mathrm{e}^{x} & \text{for } 0 < \xi < x < 1, \end{cases}$$

is the Green's function for this boundary value problem.

(ii) [4 marks] Show that the Green's function $G(x, \xi)$ for the full adjoint operator L^* and the Green's function $g(x, \xi)$ for L satisfy $G(x, \xi) = g(\xi, x)$.

(iii) [7 marks] By forming an inner product of $Ly(x) = f(x)$ with $G(x, \xi)$, show that the solution y of (7) is given by

$$y(\xi) = B(\alpha, \beta, \xi) + \int_0^1 f(x) g(\xi, x)\mathrm{d}x,$$

where you should explicitly determine the expression $B(\alpha, \beta, \xi)$, which only depends on α, β and ξ.

(b) [8 marks] State what it means for a sequence of distributions u_N, $N = 1, 2, \ldots$ *to converge* to another distribution u as $N \to \infty$.

For integer $N > 0$, consider the three cases

$$\text{(i) } R_N = \sum_{n=1}^{N} \delta\left(x - n\right), \quad \text{(ii) } S_N = \sum_{n=1}^{N} \delta\left(x - \frac{1}{n}\right), \quad \text{(iii) } T_N = \sum_{n=1}^{N} \frac{\delta\left(x - \frac{1}{n}\right)}{n^2},$$

in turn and determine for each of the sequences R_N, S_N, T_N whether or not it converges in the distributional sense, giving reasons for your statement.