Question 1: Part (a) is Bookwork. Parts (b), (c), (d) are Similar to seen results for plane polars.

Question 2: Parts (a) and (b) are Bookwork for $\mathbf{F}$, and Similar for M. Part (c) is Bookwork (but logically belongs here). Part (d) is New.

Question 3: Part (a) is Bookwork, part (b) is New, but straightforward. Part (c) is (very) Similar to seen results for water waves. Part (d) is New, but partly Similar to seen separable solutions.

1. (a) [5 marks] The components of $\mathbf{u} = \nabla \times (\psi \mathbf{k})$ give $u = \partial_y \psi$ and $v = -\psi_x$, so $\psi$ is given by

$$\psi(\boldsymbol{x}) = \psi_0 + \int_0^{\boldsymbol{x}} (u\,\mathrm{d}y - v\,\mathrm{d}x),$$

with $\psi_0$ an arbitrary constant. The difference between the integrals along two different curves is

$$\int_{\mathcal{C}_1} (u\,\mathrm{d}y - v\,\mathrm{d}x) - \int_{\mathcal{C}_2} (u\,\mathrm{d}y - v\,\mathrm{d}x) = \int_{\mathcal{C}_1 - \mathcal{C}_2} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \iint_{\mathcal{S}} \nabla\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}y = 0,$$

by the 2D divergence (or Green's) theorem, where $\mathcal{S}$ is the surface bounded by the closed curve $\mathcal{C}_1 - \mathcal{C}_2$, and $\nabla\cdot\mathbf{u} = \nabla\cdot(\nabla\times(\psi\mathbf{k})) \equiv 0$.

(b) [3 marks] Using the given formulae,

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = 0,$$

and $\nabla\times\mathbf{u} = 0$ is equivalent to

$$\begin{aligned} 0 &= \frac{\partial}{\partial r}(r u_\theta) - \frac{\partial}{\partial\theta}(u_r) = \frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\Psi_r\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\Psi_\theta\right), \\ &= -\frac{1}{r^2\sin\theta}\left(r^2\Psi_{rr} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\Psi_\theta\right)\right). \end{aligned}$$

(c) [11 marks] The given expression for $\widetilde{\Psi}(r,\theta)$ vanishes on $r = a$, which is a streamline. We need to check the flow is still a potential flow.
Consider $\hat{\Psi} = (r/a)\Psi(a^2/r,\theta)$, with derivatives

$$\partial_r\hat{\Psi} = (1/a)\Psi(a^2/r,\theta) - (a/r)\hat{\Psi}'(a^2/r,\theta), \quad \partial_{rr}\hat{\Psi} = (a^3/r^2)\Psi''(a^2/r,\theta),$$

where $'$ denotes derivative with respect to the first argument.
Substituting into part (b),

$$\begin{aligned} r^2\hat{\Psi}_{rr} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\hat{\Psi}_\theta\right) &= \frac{a^3}{r}\Psi''(a^2/r,\theta) + \frac{r}{a}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\Psi_\theta\right), \\ &= \frac{r}{a}\left[(a^2/r)^2\Psi''(a^2/r,\theta) + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\Psi_\theta\right)\right], \\ &= \frac{r}{a}\left[R^2\Psi''(R,\theta) + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\Psi_\theta\right)\right], \end{aligned}$$

with $R = a^2/r$. Hence $\hat{\Psi}$ gives a potential flow if $\Psi$ does. No new singularities appear in $r > a$ because $a^2/r < a$, and $\Psi$ has no singularities when its first argument is less than $a$. Given $|\Psi(r,\theta)| \leqslant c\,r$ for $r$ sufficiently small, $(r/a)|\Psi(a^2/r,\theta)| \leqslant c(r/a)(a^2/r) = ac$ for $r$ sufficiently large, so $\hat{\Psi}$ is bounded at large $r$, and its gradient goes to zero.

(d) [6 marks] The uniform flow $\mathbf{u} = U\mathbf{e}_z = u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta$ has components $u_r = U\cos\theta$ and $u_\theta = -U\sin\theta$ in the given spherical coordinates.

$$u_\theta = -U\sin\theta = -\frac{1}{r\sin\theta}\Psi_r \text{ integrates to } \Psi = \frac{1}{2}Ur^2\sin^2\theta.$$

Using part (c), the Stokes streamfunction is

$$\widetilde{\Psi} = \frac{1}{2}U\sin^2\theta\left(r^2 - (r/a)(a^2/r)^2\right) = \frac{1}{2}Ur^2\sin^2\theta\left(1-(a/r)^3\right).$$

The velocity components are

$$u_r = U\cos\theta\left(1-(a/r)^3\right), \quad u_\theta = -U\sin\theta\left(1+(1/2)(a/r)^3\right).$$

The $u_r$ component integrates to give

$$\phi = Ur\cos\theta\left(1+(1/2)(a/r)^3\right) + f(\theta),$$

and imposing $(1/r)\partial\phi/\partial\theta = u_\theta$ establishes that $f(\theta) = 0$.

2. (a) [5 marks] The force $\mathbf{F} = -\int_C p\,\mathbf{n}\,ds = -\int_C p\,(dy, -dx)$.
The moment of the inward pressure force is

$$M = \mathbf{e}_z\cdot\int \boldsymbol{x}\times(-p\mathbf{n})ds = \mathbf{e}_z\cdot\int p\boldsymbol{x}\times(-dy, dx, 0) = \mathbf{e}_z\cdot\int p\begin{vmatrix}\mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z}\\ x & y & 0\\ -dy & dx & 0\end{vmatrix} = \int p\,(xdx+ydy).$$

(b) [7 marks] By Bernoulli's theorem for steady irrotational flow,

$$p = p_0 - (1/2)\rho|\mathbf{u}|^2 = p_0 - (1/2)\rho\,|dw/dz|^2.$$

The contributions to the integrals below from the constant background pressure $p_0$ all vanish.
Since $dx - \mathrm{i}dy = d\overline{z}$, so $dy + \mathrm{i}dx = \mathrm{i}d\overline{z}$,

$$\begin{aligned} F_x - \mathrm{i}F_y &= \frac{1}{2}\mathrm{i}\rho\int_c\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 d\overline{z} = \frac{1}{2}\mathrm{i}\rho\int_c\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}d\overline{z} = \frac{1}{2}\mathrm{i}\rho\int_c\frac{\mathrm{d}w}{\mathrm{d}z}d\overline{w},\\ &= \frac{1}{2}\mathrm{i}\rho\int_c\frac{\mathrm{d}w}{\mathrm{d}z}dw, \text{ since } d\overline{w} = dw \text{ on } C \text{ as } \psi = \mathrm{Im}\,w = \text{constant},\\ &= \frac{1}{2}\mathrm{i}\rho\int_c\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 dz. \end{aligned}$$

Similarly, $zd\overline{z} = (x+\mathrm{i}y)(dx-\mathrm{i}dy) = xdx + ydy + \mathrm{i}(ydx - xdy)$ so

$$\begin{aligned} M &= -\frac{1}{2}\rho\int_c\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2\mathrm{Re}\,(zd\overline{z}) = -\frac{1}{2}\rho\mathrm{Re}\int_c\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 zd\overline{z} = -\frac{1}{2}\rho\mathrm{Re}\int_c z\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}d\overline{z},\\ &= -\frac{1}{2}\rho\mathrm{Re}\int_c z\frac{\mathrm{d}w}{\mathrm{d}z}d\overline{w} = -\frac{1}{2}\rho\mathrm{Re}\int_c z\frac{\mathrm{d}w}{\mathrm{d}z}dw = -\frac{1}{2}\rho\mathrm{Re}\int_c z\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 dz. \end{aligned}$$

(c) [4 marks] Using the method of images with an image source $Q$ at $-\mathrm{i}a$,

$$w = \frac{Q}{2\pi}\log(z-\mathrm{i}a) + \frac{Q}{2\pi}\log(z+\mathrm{i}a),$$

and

$$u - \mathrm{i}v = \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{Q}{2\pi}\left(\frac{1}{z-\mathrm{i}a} + \frac{1}{z+\mathrm{i}a}\right) = \frac{Q}{\pi}\frac{z}{z^2+a^2}.$$

The RHS is purely real (so $v = 0$) on the boundary $z = x$ is real. This is the no-flux boundary condition on the wall.

(d) [9 marks] Taking the integral along a large part of the boundary wall, $z \in [-R, R]$, and closing with a large semicircle in the upper half-plane,

$$\begin{aligned} F_x - \mathrm{i}F_y &= \frac{1}{2}\mathrm{i}\rho \int_{c'} \frac{Q^2}{4\pi^2} \left( \frac{1}{z - \mathrm{i}a} + \frac{1}{z + \mathrm{i}a} \right)^2 dz, \\ &= \frac{1}{2}\mathrm{i}\rho \frac{Q^2}{4\pi^2} \int_{c'} 2 \frac{1}{z - \mathrm{i}a} \frac{1}{z + \mathrm{i}a} + \cdots \, dz, \\ &= \frac{1}{2}\mathrm{i}\rho \frac{Q^2}{4\pi^2} 2\pi\mathrm{i} \frac{2}{2\mathrm{i}a} = \mathrm{i}\frac{\rho Q^2}{4\pi a}. \end{aligned}$$

The integrand is $O(1/R^2)$ on the large semicircle, so its contribution vanishes. Hence $F_x = 0$, and $F_y = -\rho Q^2/(4\pi a)$ is directed downwards,
The corresponding moment integral is

$$M = -\frac{1}{2}\rho \,\mathrm{Re} \int_C \frac{Q^2}{\pi^2} z \left( \frac{z}{z^2 + a^2} \right)^2 dz.$$

Now the contribution from a large semicircle is finite, as the integrand is $O(1/R)$. It is easier to use the oddness of the integrand for $z = x$ on the real axis to argue that $M = 0$ (as expected from symmetry). Restricting the integration to the symmetric intervals $[-R, R]$ gives the principal part of the otherwise divergent integral. (Nothing on this last point is expected.)

3. (a) [6 marks] Reynolds' transport theorem says, for any material volume $V(t)$ and (continuously differentiable) function $f(\boldsymbol{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} f \,\mathrm{d}V = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla\cdot(f\mathbf{u}) \,\mathrm{d}V.$$

The mass inside a material volume is constant, so

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho \,\mathrm{d}V = \iiint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) \,\mathrm{d}V.$$

This holds for all material volumes, so $\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0$.
Writing $f = \rho h$ for any $h(\boldsymbol{x}, t)$ establishes the corollary of Reynolds' transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho h \,\mathrm{d}V = \iiint_{V(t)} \frac{\partial(\rho h)}{\partial t} + \nabla\cdot(\rho h\mathbf{u}) \,\mathrm{d}V = \iiint_{V(t)} \rho \frac{\mathrm{D}h}{\mathrm{D}t} + h \underbrace{(\partial_t \rho + \nabla\cdot(\rho\mathbf{u}))}_{=0} \,\mathrm{d}V.$$

In an inviscid fluid with no body force, the only force exerted on the material volume $V(t)$ is an inward pressure at the boundary. Newton's second law is thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V(t)} \rho\mathbf{u} \,\mathrm{d}V = \iint_{\partial V(t)} -p\mathbf{n} \,\mathrm{d}S = -\iiint_{V(t)} \nabla p \,\mathrm{d}V,$$

by a corollary of the divergence theorem. Applying the corollary of Reynolds' transport theorem component-by-component gives

$$\iiint_{V(t)} \left( \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla p \right) \mathrm{d}V = 0.$$

This holds for all material volumes, so again the integrand vanishes pointwise.

(b) [5 marks] Writing $\rho = \rho_0 + \rho', p = p_0 + c^2\rho', \mathbf{u} = \nabla\phi$ (already small) and linearising:

$$\partial_t \rho' + \rho_0 \nabla^2 \phi = 0, \quad \rho_0 \partial_t \nabla\phi = -c^2 \nabla \rho'.$$

The second of these gives $\phi_t = -(c^2/\rho_0)\rho'$. Substituting this expression into $\partial_t$ of the first equation gives

$$\partial_{tt}\rho' = -\rho_0 \nabla^2 \phi_t = c^2 \nabla^2 \rho'.$$

The wave equation for $\phi$ follows from putting $\rho' = -(\rho_0/c^2)\phi_t$ into the line above and cancelling a $\partial_t$. Alternatively, make the same substitution to eliminate $\rho'$ between the two first order equations at the top of the page. In both cases one may add an arbitrary function of $t$ alone to $\phi$, but nothing about this is expected.

(c) [4 marks] The boundary conditions are $\mathbf{u}\cdot\mathbf{n} = \partial_y\phi = 0$ on $y = 0$ and $y = H$, and $\mathbf{u}\cdot\hat{\mathbf{x}} = \partial_x\phi = \dot{X} = \epsilon\omega\cos(\omega t)\cos(\pi y/H)$ on $x = 0$ (by linearisation).
We want $\phi$ bounded as $x \to +\infty$.

(d) [10 marks] Seeking a separable solution $\phi = \epsilon F(x)\cos(\omega t)\cos(\pi y/H)$ gives

$$-\omega^2 - c^2\left(F''/F - \pi^2/H^2\right) = 0,$$

which rearranges into

$$F''/F = \pi^2/H^2 - \omega^2/c^2.$$

We need $\phi$ to be bounded as $x \to +\infty$, so we can take the exponentially decaying solution

$$F(x) = A\exp(-\kappa x) \text{ with } \kappa = \left(\pi^2/H^2 - \omega^2/c^2\right)^{1/2},$$

when $0 < \omega < \omega_c = \pi c/H$. The boundary condition at $x = 0$ determines $A = -\omega/\kappa$, so

$$\phi = -\epsilon\frac{\omega}{\kappa}\exp(-\kappa x)\cos(\omega t)\cos(\pi y/H).$$

When $\omega > \omega_c$ we get oscillatory solutions. The general solution of the one-dimensional wave equation suggests seeking a right-going wave (i.e. a radiation condition) solution as a function of $kx - \omega t$ so we try

$$\phi = \epsilon B\sin(kx - \omega t)\cos(\pi y/H) \text{ with } k = \left(\omega^2/c^2 - \pi^2/H^2\right)^{1/2}.$$

Substituting into the boundary condition at $x = 0$ determines $B = \omega/k$.