A10 Fluids and Waves solutions – version 5 of 28/04/16

Question 1: Part (a) is Bookwork. The calculation of $\mathbf{u}$ from the flow map in (b) is New. Parts (b), (c), (d) are otherwise standard vector calculus exercises that are Similar to seen results for purely 2D flows in the xy plane.

1. (a) [5 marks]

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p, \text{ and } \nabla\cdot\mathbf{u} = 0,$$

where the density ρ is constant. Use a vector identity to rewrite as

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) = -\nabla\left(\frac{p}{\rho}\right)$$

and take the curl to eliminate the two gradient terms, leaving $\dfrac{\partial \boldsymbol{\omega}}{\partial t} + \nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = \mathbf{0}$.

(b) [8 marks] The Eulerian velocity field is $\mathbf{u} = \left.\dfrac{\mathrm{D}\mathbf{x}}{\mathrm{D}t}\right|_{\mathbf{X}}$, with components

$$u = \left.\frac{\mathrm{D}x}{\mathrm{D}t}\right|_{\mathbf{X}} = -\alpha x + \mathrm{e}^{-\alpha t}\left[-2X\Omega\mathrm{e}^{2\alpha t}\sin\left(\frac{\Omega}{\alpha}(\mathrm{e}^{2\alpha t}-1)\right) - 2Y\Omega\mathrm{e}^{2\alpha t}\cos\left(\frac{\Omega}{\alpha}(\mathrm{e}^{2\alpha t}-1)\right)\right],$$
$$= -\alpha x - 2\Omega y\mathrm{e}^{2\alpha t},$$
$$v = \left.\frac{\mathrm{D}y}{\mathrm{D}t}\right|_{\mathbf{X}} = -\alpha y + \mathrm{e}^{-\alpha t}\left[-2Y\Omega\mathrm{e}^{2\alpha t}\sin\left(\frac{\Omega}{\alpha}(\mathrm{e}^{2\alpha t}-1)\right) + 2X\Omega\mathrm{e}^{2\alpha t}\cos\left(\frac{\Omega}{\alpha}(\mathrm{e}^{2\alpha t}-1)\right)\right],$$
$$= -\alpha x + 2\Omega x\mathrm{e}^{2\alpha t},$$
$$w = \left.\frac{\mathrm{D}z}{\mathrm{D}t}\right|_{\mathbf{X}} = Z\mathrm{e}^{2\alpha t} = 2\alpha z.$$

This gives $\mathbf{u} = (-\alpha x - 2\Omega y\mathrm{e}^{2\alpha t}, -\alpha y + 2\Omega x\mathrm{e}^{2\alpha t}, 2\alpha z)$ as required. Now we can calculate

$$\nabla\cdot\mathbf{u} = -\alpha - \alpha + 2\alpha = 0.$$

Similarly, the vorticity is

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -\alpha x - 2\Omega y\mathrm{e}^{2\alpha t} & -\alpha y + 2\Omega x\mathrm{e}^{2\alpha t} & 2\alpha z \end{vmatrix} = (0, 0, 4\Omega\mathrm{e}^{2\alpha t}).$$

(c) [8 marks]

$$\boldsymbol{\omega}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4\Omega\mathrm{e}^{2\alpha t} \\ u & v & w \end{vmatrix} = 4\Omega\mathrm{e}^{2\alpha t}\,(-v, u, 0),$$

and

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = 4\Omega\mathrm{e}^{2\alpha t}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -v & u & 0 \end{vmatrix} = 4\Omega\mathrm{e}^{2\alpha t}(0, 0, \partial_x u + \partial_y v) = 4\Omega\mathrm{e}^{2\alpha t}(0, 0, -2\alpha).$$

Hence

$$\nabla\times(\boldsymbol{\omega}\times\mathbf{u}) = -8\Omega\alpha\mathrm{e}^{2\alpha t}\mathbf{k} = -\frac{\partial\boldsymbol{\omega}}{\partial t}.$$

(d) [4 marks] A radial inflow $(-\alpha x, -\alpha y, 0)$ and compensating outflow parallel to the z axis is superimposed upon a rigid rotation. The vorticity $4\Omega\mathrm{e}^{2\alpha t}$ increases over time to conserve the angular momentum of fluid particles as they move towards the z axis. This is an example of vortex stretching.

Question 2 - part (a) has appeared twice in previous exams (including last year) and on this year's problem sheets. Part (b) is bookwork - the canonical application of the Milne-Thomson circle theorem. Part (c) is seen in lectures. Part (d) is Similar to lectures. They have seen that it is easier to transform to an integral in z over $|z| = a$, as suggested by the hint, rather than an integral round the ellipse in ζ, but not this particular calculation. The small piece of interpretation at the end is New.

2. (a) [5 marks] The moment of the inward pressure force is

$$M = \mathbf{e}_z \cdot \int \mathbf{x} \times (-p\mathbf{n}) \mathrm{d}s = \int p\,(x\mathrm{d}x + y\mathrm{d}y).$$

By Bernoulli's theorem for steady irrotational flow,

$$p = p_0 - \frac{1}{2}\rho|\mathbf{u}|^2 = p_0 - \frac{1}{2}\rho\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2.$$

The contributions to the integrals below from the constant background pressure p_0 vanish. Using $z\mathrm{d}\overline{z} = (x + \mathrm{i}y)(\mathrm{d}x - \mathrm{i}\mathrm{d}y) = x\mathrm{d}x + y\mathrm{d}y + \mathrm{i}(y\mathrm{d}x - x\mathrm{d}y)$,

$$M = -\frac{1}{2}\rho\int_c\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \mathrm{Re}\,(z\mathrm{d}\overline{z}) = -\frac{1}{2}\rho\mathrm{Re}\int_c\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 z\mathrm{d}\overline{z} = -\frac{1}{2}\rho\mathrm{Re}\int_c z\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{z}}\mathrm{d}\overline{z},$$
$$= -\frac{1}{2}\rho\mathrm{Re}\int_c z\frac{\mathrm{d}w}{\mathrm{d}z}d\overline{w} = -\frac{1}{2}\rho\mathrm{Re}\int_c z\frac{\mathrm{d}w}{\mathrm{d}z}\mathrm{d}w = -\frac{1}{2}\rho\mathrm{Re}\int_c z\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2\mathrm{d}z.$$

(b) [4 marks] Putting $z = a\mathrm{e}^{\mathrm{i}\theta}$ to parametrise the circle, the complex potential becomes

$$w = U\left(a\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\alpha} + a\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\alpha}\right) = Ua\mathrm{e}^{\mathrm{i}(\theta-\alpha)} + \text{complex conjugate}$$

so the streamfunction $\psi = \mathrm{Im}\,w$ is constant on $|z| = a$. Far from the cylinder, the flow approaches the uniform flow $w \sim Uz\mathrm{e}^{-\mathrm{i}\alpha}$ inclined at angle α to the positive real axis.

(c) [5 marks] Putting $z = a\mathrm{e}^{\mathrm{i}\theta}$ again gives

$$\zeta = z + c^2/z = a\mathrm{e}^{\mathrm{i}\theta} + \frac{c^2}{a}\mathrm{e}^{-\mathrm{i}\theta} = \left(a + \frac{c^2}{a}\right)\cos\theta + \mathrm{i}\left(a - \frac{c^2}{a}\right)\sin\theta,$$

which is the parametric form of the ellipse

$$\left(\frac{\mathrm{Re}\,\zeta}{a + c^2/a}\right)^2 + \left(\frac{\mathrm{Im}\,\zeta}{a - c^2/a}\right)^2 = 1.$$

The inverse Joukowski transformation is

$$z = \zeta/2 + \left(\zeta^2/4 - c^2\right)^{1/2},$$

with the branch of $(\zeta^2/4 - c^2)^{1/2}$ chosen so that $z \sim \zeta$ as $|\zeta| \to \infty$. The flow far from the ellipse thus approaches the same uniform stream as in part (b). Its complex potential is

$$W(\zeta) = U\left[\left(\zeta/2 + \left(\zeta^2/4 - c^2\right)^{1/2}\right)\mathrm{e}^{-\mathrm{i}\alpha} + a^2\mathrm{e}^{\mathrm{i}\alpha}\left(\zeta/2 + \left(\zeta^2/4 - c^2\right)^{1/2}\right)^{-1}\right].$$

(d) [11 marks] The moment is

$$M = -\frac{1}{2}\rho \,\mathrm{Re} \int_{\text{ellipse}} \zeta \left(\frac{\mathrm{d}W}{\mathrm{d}\zeta}\right)^2 \mathrm{d}\zeta,$$
$$= -\frac{1}{2}\rho \,\mathrm{Re} \int_{|z|=a|} \zeta \left(\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^2 \frac{\mathrm{d}\zeta}{\mathrm{d}z}\,\mathrm{d}z,$$
$$= -\frac{1}{2}\rho \,\mathrm{Re} \int_{|z|=a|} \zeta \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^{-1} \mathrm{d}z,$$
$$= -\frac{1}{2}\rho U^2 \,\mathrm{Re} \int_{|z|=a|} z\,\frac{z^2+c^2}{z^2-c^2}\left(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{a^2}{z^2}\mathrm{e}^{\mathrm{i}\alpha}\right)^2 \mathrm{d}z.$$

The quicker way to evaluate this integral finds the coefficient of $1/z$ in the Laurent expansion about $z = 0$ of the integrand that holds in an annulus containing $|z| = a > c$. This is given by writing the integrand as

$$z\frac{z^2+c^2}{z^2(1-c^2/z^2)}\left(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{a^2}{z^2}\mathrm{e}^{\mathrm{i}\alpha}\right)^2 = \left(z + \frac{c^2}{z}\right)\left(1 + \frac{c^2}{z^2} - \frac{c^4}{z^4} + \cdots\right)\left(\mathrm{e}^{-2\mathrm{i}\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}\mathrm{e}^{2\mathrm{i}\alpha}\right).$$

The coefficient of $1/z$ is

$$c^2\mathrm{e}^{-2\mathrm{i}\alpha} - 2a^2 + c^2\mathrm{e}^{-2\mathrm{i}\alpha} = 2c^2\mathrm{e}^{-2\mathrm{i}\alpha} - 2a^2, \qquad (\dagger)$$

so the moment integral is

$$M = -\frac{1}{2}\rho U^2 \,\mathrm{Re}\,\left(2\pi\mathrm{i}\,(2c^2\mathrm{e}^{-2\mathrm{i}\alpha} - 2a^2)\right) = -2\pi\rho U^2 c^2 \sin 2\alpha.$$

Alternatively: The integrand has poles at $z = 0$ and $z = \pm c$. The residues at $z = \pm c$ are both equal to

$$c\frac{2c^2}{2c}\left(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{a^2}{c^2}\mathrm{e}^{\mathrm{i}\alpha}\right)^2.$$

The residue at $z = 0$ comes from writing the integrand as

$$-z\left(1+\frac{z^2}{c^2}\right)\left(1-\frac{z^2}{c^2}\right)^{-1}\left(\mathrm{e}^{\mathrm{i}\alpha} - \frac{a^2}{z^2}\mathrm{e}^{\mathrm{i}\alpha}\right)^2 = -z\left(1 + 2\frac{z^2}{c^2} + \cdots\right)\left(\mathrm{e}^{-2\mathrm{i}\alpha} - 2\frac{a^2}{z^2} + \frac{a^4}{z^4}\mathrm{e}^{2\mathrm{i}\alpha}\right).$$

The coefficient of $1/z$ in this expansion (valid for $|z| < c$) is

$$2a^2 - 2\frac{a^4}{c^2}\mathrm{e}^{2\mathrm{i}\alpha},$$

so the sum of the three residues is

$$2a^2 - 2\frac{a^4}{c^2}\mathrm{e}^{2\mathrm{i}\alpha} + 2c^2\left(\mathrm{e}^{-2\mathrm{i}\alpha} - 2\frac{a^2}{c^2} + \frac{a^4}{c^4}\mathrm{e}^{\mathrm{i}\alpha}\right) = 2c^2\mathrm{e}^{-2\mathrm{i}\alpha} - 2a^2.$$

This is the same as the coefficient of $1/z$ in the earlier Laurent expansion valid in an annulus containing $|z| = a > c$, the expression $(\dagger)$ above.

Due to the $-\sin 2\alpha$ dependence, the moment tends to align the ellipse broad side on to the oncoming stream.

3. Question 3: All Similar to a problem sheet question.

(a) [6 marks] Looking for a solution with u_θ just a function of r, the given curl formula becomes

$$\nabla\times\mathbf{u} = \frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\,\mathbf{e}_z,$$

so the irrotational flow in $r \geqslant a$ must have $u_\theta = C/r$ for some constant C. Rigid rotation with angular velocity Ω implies $u_\theta = \Omega r$ for $0 \leqslant r < a$. Making the velocity continuous at $r = a$ determines $C = \Omega a^2$, and $u_\theta = \Omega a^2/r$ as required.

(b) [6 marks] Bernoulli's theorem for steady irrotational flow (valid in $r \geqslant a$) gives

$$\frac{p}{\rho} + \frac{1}{2}|\mathbf{u}|^2 + \chi = \text{constant},$$

where $\chi = gz$ is the potential for $\mathbf{g} = -\nabla\chi = -g\,\mathbf{e}_z$.
Using $p = p_{\text{atm}}$ is constant on the free surface, and $\eta \to 0$ as $r \to \infty$, gives

$$\frac{1}{2}|\mathbf{u}|^2 + g\eta = 0, \qquad \Longrightarrow \qquad \eta = -\frac{|\mathbf{u}|^2}{2g} = -\frac{\Omega^2 a^4}{2gr^2}.$$

(c) [9 marks] The relevant forms of the components of the Euler equations are

$$-\frac{u_\theta^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \text{ and } \frac{1}{\rho}\frac{\partial p}{\partial z} = -g.$$

Putting $u_\theta = \Omega r$, these integrate to give

$$\frac{p}{\rho} = \frac{1}{2}\Omega^2 r^2 + f(z), \text{ and } \frac{p}{\rho} = -gz + h(r),$$

where $f(z)$ and $h(r)$ are two functions of integration. Making the two expressions consistent determines

$$\frac{p}{\rho} = \frac{1}{2}\Omega^2 r^2 - gz + C,$$

where C is a constant. Making the pressure on the free surface continuous at $r = a$ gives

$$\frac{p_{\text{atm}}}{\rho} = \frac{1}{2}\Omega^2 a^2 - g\left(-\frac{\Omega^2 a^2}{2g}\right) + C, \qquad \Longrightarrow \qquad C = \frac{p_{\text{atm}}}{\rho} - \Omega^2 a^2,$$

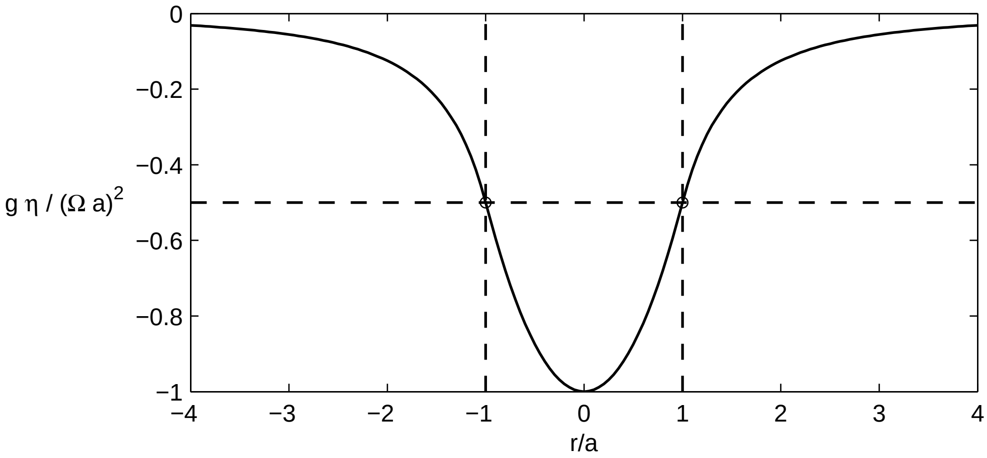
so

$$\frac{p}{\rho} = \frac{p_{\text{atm}}}{\rho} + \frac{1}{2}\Omega^2 r^2 - gz - \Omega^2 a^2.$$

Putting $p = p_{\text{atm}}$ on $z = \eta$ determines the free suface location in $r < a$:

$$\eta = \frac{\Omega^2 a^2}{g}\left(\frac{r^2}{2a^2} - 1\right).$$

(d) [4 marks] Putting the calculated free surface positions in $r < a$ and $r \geqslant a$ together gives:



The free surface position and its derivative are both continuous at $r = a$.