Q1(a) Euler's equations for an inviscid, incompressible fluid in 3-D are

$$\underline{u}_t + (\underline{u}.\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \underline{g} \quad (1)$$

$$\underline{\nabla}.\underline{u} = 0 \quad (2).$$

Vorticity is defined as : $\underline{\omega} = \underline{\nabla}\wedge\underline{u}$

Using the vector identity :

$$(\underline{u}.\underline{\nabla})\underline{u} = \underline{\nabla}\tfrac{1}{2}\underline{u}^2 - \underline{u}\wedge\underline{\omega}$$

For conservative body forces : $\underline{g} = -\underline{\nabla}\chi$

$$\therefore \quad \underline{u}_t + (\underline{u}.\underline{\nabla})\underline{u} = \underline{u}_t + \underline{\nabla}\tfrac{1}{2}|\underline{u}|^2 - \underline{u}\wedge\underline{\omega}$$

$$= -\underline{\nabla}\left(\frac{p}{\rho} + \chi\right)$$

$$\therefore \quad \underline{u}_t - \underline{u}\wedge\underline{\omega} = -\underline{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \chi\right) \quad (3)$$

For the vorticity eqn, take $\underline{\nabla}\wedge(3)$ to get

$$(\underline{\nabla}\wedge\underline{u})_t - \underline{\nabla}\wedge(\underline{u}\wedge\underline{\omega}) = -\underline{\nabla}\wedge\underline{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \chi\right)$$

$$= 0$$

Now $\underline{\nabla}\wedge(\underline{u}\wedge\underline{\omega}) = \underline{e}_r\frac{\partial}{\partial x_r}\wedge(\underline{u}\wedge\underline{\omega})$

$$= \underline{e}_r\wedge\left(\frac{\partial \underline{u}}{\partial x_r}\wedge\underline{\omega}\right) + \underline{e}_r\wedge\left(\underline{u}\wedge\frac{\partial \underline{\omega}}{\partial x_r}\right)$$

$$= (\underline{e}_r.\underline{\omega})\frac{\partial \underline{u}}{\partial x_r} - \left(\underline{e}_r.\frac{\partial \underline{u}}{\partial x_r}\right)\underline{\omega}$$

$$+ \left(\underline{e}_r.\frac{\partial \underline{\omega}}{\partial x_r}\right)\underline{u} - (\underline{e}_r.\underline{u})\frac{\partial \underline{\omega}}{\partial x_r}$$

$$\therefore \nabla \wedge (\underline{u} \wedge \underline{\omega}) = (\underline{\omega} \cdot \underline{e}_r)\frac{\partial \underline{u}}{\partial x_r} - (\nabla \cdot \underline{u})\,\underline{\omega} + (\nabla \cdot \underline{\omega})\,\underline{u} - \left(\underline{u} \cdot \underline{e}_r\right)\frac{\partial \underline{\omega}}{\partial x_r}$$

By (2) $\nabla \cdot \underline{u} = 0$, since $\underline{\omega} = \nabla \wedge \underline{u}$, $\nabla \cdot (\nabla \wedge \underline{u}) = 0$

$$\therefore \nabla \wedge (\underline{u} \wedge \underline{\omega}) = (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega}$$

$\therefore$ vorticity eqn becomes

$$\underline{\omega}_t = (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega}$$

ie
$$\frac{D\underline{\omega}}{Dt} = \underline{\omega}_t + (\underline{u} \cdot \nabla)\underline{\omega} = (\underline{\omega} \cdot \nabla)\underline{u} \qquad (4)$$

(b) For 2-D flows with no body forces, & $\underline{u} = \underline{u}(x, y)$

$$\nabla \cdot \underline{u} = u_x + v_y$$

$$\therefore \nabla \wedge \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = (0, 0, v_x - u_y) \qquad (5)$$

From (2) we define the streamfunction ψ, satisfying

$$u = \psi_y, \quad v = -\psi_x$$

so that $\nabla \wedge \underline{u} = (0, 0, -\nabla^2 \psi) = \underline{\omega}$

$\therefore$ writing $\underline{\omega} = (0, 0, \zeta)$, we obtain

$$\nabla^2 \psi = -\zeta \qquad (6)$$

In components, (4) becomes

$$(0,0,\zeta_t) + (u,v,0)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\underline{\omega}$$

$$= (0,0,\zeta)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)(\underline{u})$$

$$\therefore \zeta_t\underline{k} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(0,0,\zeta)$$

$$= 0 \qquad \left[\text{since } \zeta\frac{\partial}{\partial z}\underline{u} = \right.$$

$$\therefore \zeta_t\underline{k} + \left(\psi_y\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right) = 0$$

$$\therefore \qquad \frac{\partial(\zeta,\psi)}{\partial(x,y)}$$

$$\therefore \frac{\partial\zeta}{\partial t} = -\frac{\partial(\zeta,\psi)}{\partial(x,y)} \qquad (7)$$

(c)(i) For a stream in the x-direction with constant vorticity ζ_c & mean velocity U,

(6) $\Rightarrow$ $\psi_{xx} + \psi_{yy} = -\zeta_c$

Since $\underline{u}(x,y)$ is in the x-direction &, independent of x, $u = u(y)$

$$\therefore \zeta_c = -\psi_{yy}$$

$\int_y$ $u = \psi_y = -\zeta_c y + A$ $\Rightarrow$ $A = U$

since $u = U$ on $y = 0$

$$\psi_y = u = U - \zeta_c y$$

$\int_y$ $\therefore \psi = Uy - \frac{1}{2}\zeta_c y^2 + B$ & on $y=0$ $\psi = B$ so absorb B into ψ

$\psi = Uy - \frac{1}{2}\zeta_c$

1(c)(ii) $u = \psi_y = U - \varsigma_c y$, $v = 0$

$\underline{u}_t = 0$, $\underline{\omega} = (0, 0, \varsigma_c)$

$$\therefore \underline{u} \wedge \underline{\omega} = \begin{vmatrix} \underline{\hat{i}} & \underline{\hat{j}} & \underline{\hat{k}} \\ u & 0 & 0 \\ 0 & 0 & \varsigma_c \end{vmatrix} = -u\varsigma_c \underline{\hat{j}}$$

From eqn (3): $u\varsigma_c \underline{\hat{j}} = -\underline{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \chi\right)$

$\therefore \underline{\hat{j}}$: $u\varsigma_c = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \chi\right)$.

Since $\underline{g}$ is in the z-direction, $\frac{\partial \chi}{\partial y} = 0$

$\frac{1}{2}|\underline{u}|^2 = \frac{1}{2}(U - \varsigma_c y)^2 = \frac{1}{2}(U^2 - 2U\varsigma_c y + \varsigma_c^2 y^2)$

$$\therefore U\varsigma_c = -\frac{1}{\rho}\frac{dp}{dy} - \frac{1}{2}(-2U\varsigma_c + 2\varsigma_c y)$$

$$= -\frac{1}{\rho}\frac{dp}{dy} + U\varsigma_c - \varsigma_c y$$

$$\therefore \frac{dp}{dy} = -\rho\varsigma_c y$$

$$\therefore p = -\rho\varsigma_c \frac{y^2}{2} + P_0 \qquad [4]$$

2(a)

(i) Volume flux $\Phi = \oint_C \underline{u} \cdot \underline{dr} = \oint_C u_r r \, d\theta$

B/W ($u_r = u_r(r)$ is r^m gpt) $= 2\pi r u_r$

$\therefore u_r = \frac{\partial \phi}{\partial r} = \frac{\Phi}{2\pi r}$, where ϕ is velocity potential $= \frac{\Phi}{2\pi} \ln r$

Cauchy-Riemann eqns give

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} \Rightarrow \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\Phi}{2\pi r}$$

$$\therefore \psi = \frac{\Phi}{2\pi}\theta$$

$$\therefore W = \phi + i\psi = \frac{\Phi}{2\pi}(\ln r + i\theta) = \frac{\Phi}{2\pi}\ln z$$

is complex potential for source at $z=0$

$$\therefore w(z) = \frac{\Phi}{2\pi}\ln(z-a)$$

is required $\mathbb{C}$ potential for a source, strength Φ, at $z=a$ [4]

(ii) Suppose $\exists$ source of strength Φ at $\zeta = f(z)$

Then $W(\zeta) = \frac{\Phi}{2\pi}\ln(f(z) - f(a)) = w(z)$.

By Taylor's Th^m: $= \frac{\Phi}{2\pi}\ln[f'(a)(z-a) + O(z-a)^2]$ as $z \to a$

$$\therefore w(z) \sim \frac{\Phi}{2\pi}\ln(z-a) + \underbrace{\frac{\Phi}{2\pi}\ln f'(a)}_{O(1) \text{ as } z\to a}$$

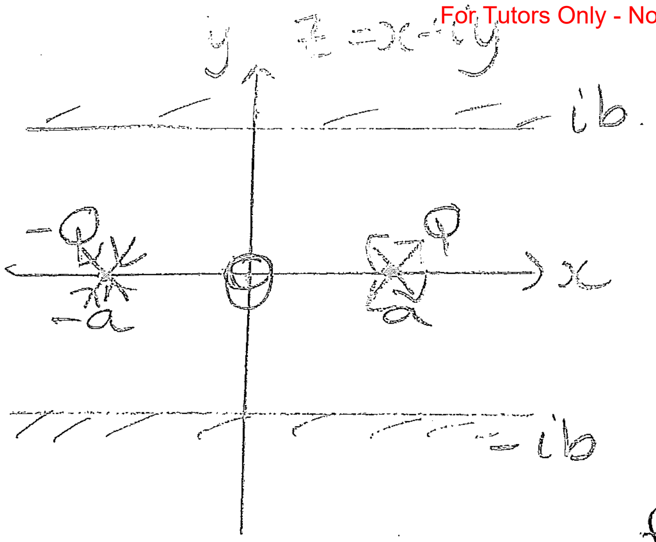
provided $f'(a) \neq 0$

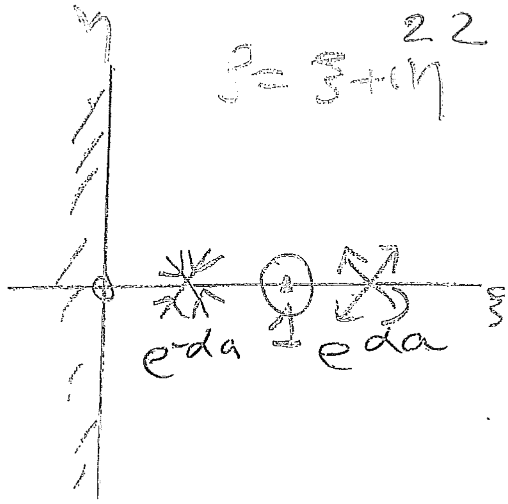
i.e. source of strength Φ at $z=a$

4

[~~6~~]

2(b)





$$\rho = e^{\alpha z}, \ \alpha > 0$$

$$\xi + i\eta = e^{\alpha x} e^{i\alpha y}$$

on $z = \pm ib$, $\xi + i\eta = e^{\alpha x} e^{\pm i\alpha b}$

Let $\alpha b = \pi/2 \Rightarrow \xi + i\eta = \pm i e^{\alpha x}$

$\left(e^{\pm i\pi/2} = \cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2} = \pm i \right)$

$\therefore$ line $y = ib$ is +ve η axis

$y = -ib$ is −ve η axis

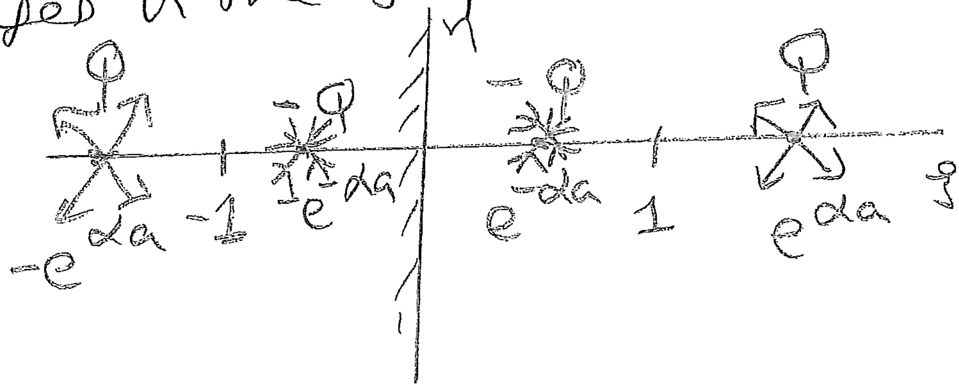
when $z = 0$, $\rho = 1 = \xi + i\eta \Rightarrow \xi = 1$

$z = a$, $e^{\alpha a} = \xi + i\eta \Rightarrow \xi = e^{\alpha a}$

$z = -a$, $e^{-\alpha a} = \xi$

Map is conformal since $\frac{d\rho}{dz} = \alpha e^{\alpha z} \neq 0$.

Complex ~~potential~~ potential for the flow, uses method of images in the ρ plane:



By method of Images:

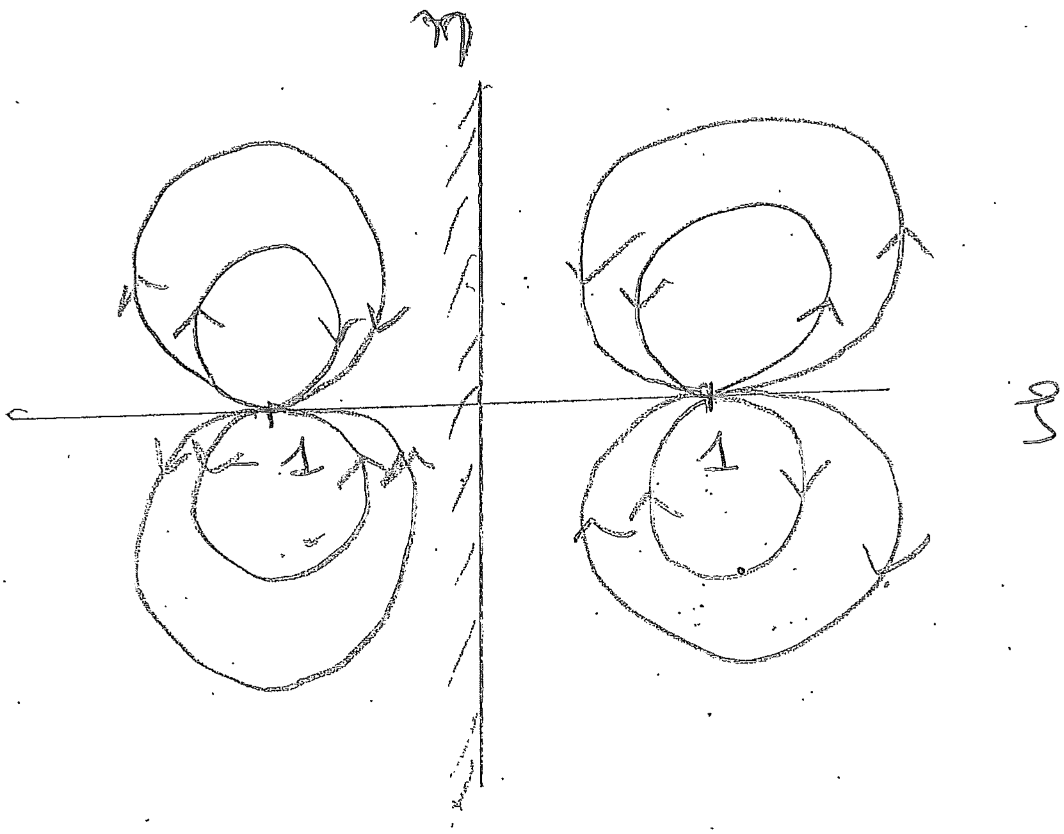
$$W(\zeta) = \frac{\Phi}{2\pi}\Big\{\ln(\zeta - e^{\alpha a}) - \ln(\zeta - e^{-\alpha a}) + \ln(\zeta + e^{\alpha a}) - \ln(\zeta + e^{-\alpha a})\Big\}$$

$$= \frac{\Phi}{2\pi}\Big\{\ln\big((\zeta - e^{\alpha a})(\zeta + e^{\alpha a})\big) - \ln\big((\zeta - e^{-\alpha a})(\zeta + e^{-\alpha a})\big)\Big\}$$

$$= \frac{\Phi}{2\pi}\ln\left(\frac{\zeta^2 - e^{2\alpha a}}{\zeta^2 - e^{-2\alpha a}}\right)$$

$$\therefore\; W(z) = \frac{\Phi}{2\pi}\ln\left(\frac{e^{2\alpha z} - e^{2\alpha a}}{e^{2\alpha z} - e^{-2\alpha a}}\right)$$

(c)



[2]

2(c) Letting $a \to 0$, $\varphi \to \infty$, the source & sink combine at $\xi = 1$ in the complex ξ-plane, to form a dipole, with an image dipole at $\xi = -1$.

$\therefore$ Expanding $e^{\pm\alpha a}$ to linear order gives

$$\frac{2\pi W_d(\xi)}{\varphi} = \left\{ \ln(\xi-(1+\alpha a)) - \ln(\xi-(1-\alpha a)) + \ln(\xi+(1+\alpha a)) - \ln(\xi+(1-\alpha a)) \right\}$$

$$= \cancel{\ln(\xi-1)} + \ln\left(1-\frac{\alpha a}{\xi-1}\right) - \cancel{\ln(\xi-1)} - \ln\left(1+\frac{\alpha a}{\xi-1}\right) + \cancel{\ln(\xi+1)} + \ln\left(1+\frac{\alpha a}{\xi+1}\right) - \cancel{\ln(\xi+1)} - \ln\left(1-\frac{\alpha a}{\xi+1}\right)$$

Expanding ln

$$\frac{2\pi W_d(\xi)}{\varphi} = \left(-\frac{\alpha a}{\xi-1} - \cancel{\frac{1}{2}\left(\frac{\alpha a}{\xi-1}\right)^2} + \ldots\right) - \left(\frac{\alpha a}{\xi-1} - \cancel{\frac{1}{2}\left(\frac{\alpha a}{\xi-1}\right)^2} + \ldots\right) + \left(\frac{\alpha a}{\xi+1} - \cancel{\frac{1}{2}\left(\frac{\alpha a}{\xi+1}\right)^2} + \ldots\right) - \left(-\frac{\alpha a}{\xi+1} - \cancel{\frac{1}{2}\left(\frac{\alpha a}{\xi+1}\right)^2} + \ldots\right)$$

$$\therefore W_d = \frac{-\cancel{2}\alpha a\varphi}{\cancel{2}\pi(\xi-1)} + \frac{\cancel{2}\alpha a\varphi}{\cancel{2}\pi(\xi+1)} = \frac{\alpha a\varphi(-2)}{\pi(\xi^2-1)}$$

Take limit $\varphi \to \infty$, $a \to 0$, & let $\mu = \dfrac{-a\varphi}{\pi}$ $\Rightarrow W_{\text{dipole}}(\xi) = \dfrac{2\alpha}{\xi^2-1}$ [4]

5.0

3 [a] Phase speed, c, is the speed of propagation of individual wave crests

B/W Wave number, k, determines the number of waves ~~passing~~ per unit distance.

Wave frequency, $\omega_r = ck$, is the number of waves per second past a given point.

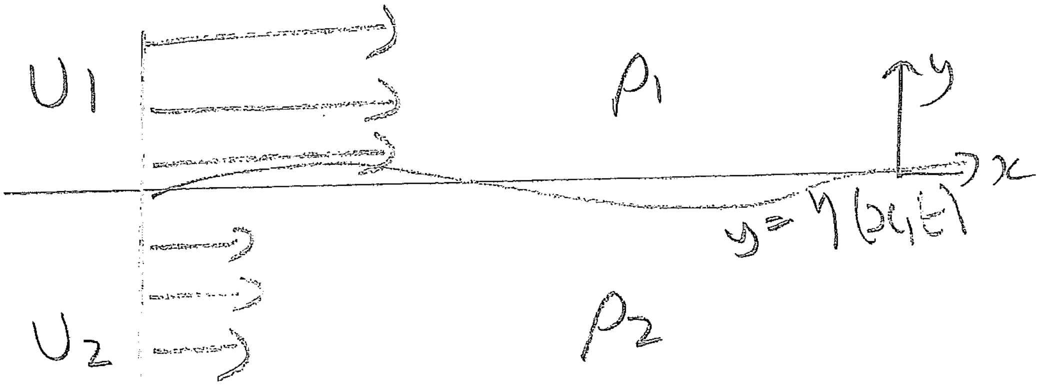
For a travelling wave $\eta = \mathbb{R}[Ae^{i(kx-\omega t)}]$

[b] Instability occurs when $\omega = \omega_r + i\omega_i$ with $\omega_i > 0$ or $kc_i > 0$.

Qn 3 3.1

B/W

Consider two inviscid fluid layers with densities ρ_1 in $y > 0$ and ρ_2 in $y < 0$, with $\rho_2 > \rho_1$. Suppose the upper fluid moves with velocity U_1 along x direction, and the lower fluid with velocity U_2 along the x-direction with $U_1 > U_2$.



The 2 fluids are incompressible

$$\therefore \quad \underline{\nabla} \cdot \underline{u}_i = 0 \qquad \checkmark \quad i = 1, 2 \qquad (1)$$

irrotational: $\underline{\nabla} \wedge \underline{u}_i = 0 \quad (i = 1, 2) \quad (2)$

$(2) \Rightarrow \exists \ \Phi_i$ st $\underline{u}_i = \nabla \Phi_i$

$\therefore (1) \Rightarrow \nabla^2 \Phi_i = 0 \qquad (3) \quad \checkmark$

subject to $\quad \Phi_1 \to 0$ as $y \to +\infty$

$\qquad \Phi_2 \to 0$ as $y \to -\infty \quad \checkmark$

We take perturbations about shear flow:

$$\Phi_i = U_i x + \phi_i$$

[4] st $\quad \phi_1 \to 0$ as $y \to +\infty \qquad (3)$

$\qquad \phi_2 \to 0$ as $y \to -\infty$

$(3) \Rightarrow \nabla^2 \phi_i = 0 \qquad (4)$

At the interface, fluid particles move with the interface. For upper fluid:

$$\therefore \frac{d}{dt}(y_+ - \eta) = 0$$

$$\therefore \left[\frac{\partial}{\partial t} + (u_+, v_+)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right][y_+ - \eta] = 0$$

$$\therefore \left[\frac{\partial}{\partial t} + (U_1 + \phi_{1x}, \phi_{1y})\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\right][y_+ - \eta] = 0$$

$$\therefore -\eta_t + (U_1 + \phi_{1x})(-\eta_x) + \phi_{1y} = 0$$

Linearising about $y=0 \Rightarrow$ (by Taylor expanding)

$$-\eta_t - U_1\eta_x + \phi_{1y} = 0$$

[2] $\therefore \quad \phi_{1y} = \eta_t + U_1\eta_x \quad$ on $y=0$ (5)a

Similarly: $\phi_{2y} = \eta_t + U_2\eta_x \quad$ on $y=0$ (5)b

for lower fluid.

& ρ_i = const

Euler's eqns $\Rightarrow$ $(i=1,2)$

$$\rho_i \underline{u}_{it} + \rho_i(\underline{u}_i\cdot\underline{\nabla})\underline{u}_i = -\nabla p_i + \rho_i \underline{g}$$

$$\& \quad (\underline{u}\cdot\underline{\nabla})\underline{u} = \underline{\nabla}\tfrac{1}{2}u^2 - \underline{u}\wedge\underbrace{(\underline{\nabla}\wedge\underline{u})}_{=0}$$

Also $\underline{g} = -\nabla gz, \quad \underline{u}_i = \nabla\Phi_i$

$$\therefore \rho_i\underline{\nabla}\Phi_{it} + \rho_i\left[\underline{\nabla}\tfrac{1}{2}|\underline{u}_i|^2\right] = -\nabla(p_i + \rho_i g y)$$

$$\therefore \underline{\nabla}\left[\rho_i\phi_{it} + \rho_i\tfrac{1}{2}|\underline{u}_i|^2 + p_i + \rho_i g y\right] = 0$$

integrating gives unsteady Bernoulli's eqn:

$$\rho_i \phi_{it} + \rho_i U_i \phi_{ix} + \rho_i (\phi_{ix}^2 + \phi_{iy}^2) + \rho_i g y$$

neglect to 1st order

$$= -p_i + C_i \rho_i \quad (+\tfrac{1}{2}\rho_i U_i^2)$$

cty of pressure across interface $y = \eta$

$$\rho_i(\phi_{it} + g\eta + U_i\phi_{ix}) = -p_i + C_i + \tfrac{1}{2}U_i^2\rho_i$$

& also "$\frac{1}{2}\rho_1 U_1^2 - C_1 = \frac{1}{2}\rho_2 U_2^2 - C_2$

[4] $\Rightarrow \rho_1(\phi_{1t} + g\eta + U_1\phi_{1x}) = \rho_2(\phi_{2t} + g\eta + U_2\phi_{2x})$ on $y = 0$ (6)

Consider T.W. on the surface:

$$\eta = \mathbb{R}(Ae^{i(kx-\omega t)})$$

(5) a,b $\Rightarrow$ $\phi_j \propto e^{i(kx-\omega t)} f_j(y)$

$\nabla^2\phi_j = 0 \Rightarrow \quad f_{jyy} - k^2 f_j = 0$

$$\therefore f_j = B_j e^{ky} + C_j e^{-ky}$$

For $j=1$: $y \to \infty$, $f_1 \to 0 \Rightarrow B_1 = 0$

$j=2$: $y \to -\infty$, $f_2 \to 0 \Rightarrow C_2 = 0$

$$\therefore \phi_1 = C_1 e^{-ky} e^{i(kx-\omega t)}$$

$$\phi_2 = B_2 e^{ky} e^{i(kx-\omega t)}$$

$$\eta_t + U_j \eta_x = \phi_{j\,y}$$

$$\Rightarrow A\left[-i\omega + U_j ik\right] e^{i(kx-\omega t)} = f_j'(y)\, e^{i(kx-\omega t)}$$

$$\therefore \quad iA\left[U_1 k - \omega\right] = -kC_1 \qquad (a)$$

$$iA\left[U_2 k - \omega\right] = kB_2 \qquad (b)$$

& (6) $\Rightarrow$

$$\rho_1\left(-i\omega C_1 + gA + ikC_1U_1\right) = \rho_2\left(-i\omega B_2 + gA + ikB_2U_2\right)$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} i(U_1k-\omega) & 0 & \cancel{k} & k \\ i(U_2k-\omega) & -k & \cancel{0} & 0 \\ g(\rho_1-\rho_2) & \rho_2(i\omega - kiU_2) & & \rho_1(-i\omega + ikU_1) \end{bmatrix}}_{=D} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

For non-trivial solns: $\det D = 0$

$\Rightarrow$
$$\omega^2(\rho_1 k + \rho_2 k) + \omega\left[-2k^2(\rho_1U_1 + \rho_2U_2)\right]$$

[4] $$+ \rho_1 k^3 U_1^2 + \rho_2 k^3 U_2^2 + k^2 g(\rho_1 - \rho_2) = 0$$

Let $\omega = kc$ to get

$$c^2(\rho_1+\rho_2) - 2c(\rho_1U_1 + \rho_2U_2) + (\rho_1U_1^2 + \rho_2U_2^2) + \frac{g}{k}(\rho_1-\rho_2) = 0$$

[1] $$\therefore \quad c = \frac{\rho_1U_1 + \rho_2U_2}{\rho_1+\rho_2} \pm \frac{\Delta^{1/2}}{\rho_1+\rho_2} \quad \text{where}$$

$$\Delta = (\rho_1U_1 + \rho_2U_2)^2 - (\rho_1^2 - \rho_2^2)\, g/k$$

3(c) 3(c)

(3.5) replaces old version

$$c^2(\rho_1+\rho_2) - 2c(\rho_1 U_1 + \rho_2 U_2) + (\rho_1 U_1^2 + \rho_2 U_2^2) + \frac{g}{k}(\rho_1 - \rho_2) = 0$$

$$\therefore \quad c = \frac{2(\rho_1 U_1 + \rho_2 U_2)}{2(\rho_1+\rho_2)} \pm \frac{1}{\cancel{2}(\rho_1+\rho_2)}\left[\cancel{4}(\rho_1 U_1 + \rho_2 U_2)^2 - \cancel{4}\left[(\rho_1 U_1^2 + \rho_2 U_2^2) + \frac{4g}{k}(\rho_1 - \rho_2)\right] \times [\rho_1 + \rho_2)\right]^{1/2}$$

$$\Delta = \cancel{4}(\rho_1^2 U_1^2 + \rho_2^2 U_2^2 + 2\rho_1\rho_2 U_1 U_2) - (\rho_1+\rho_2)(\rho_1 U_1^2 + \rho_2 U_2^2) - \frac{4g}{k}(\rho_1^2 - \rho_2^2)$$

$$= \cancel{\rho_1^2 U_1^2} + \cancel{\rho_2^2 U_2^2} + 2\rho_1\rho_2 U_1 U_2 - \cancel{\rho_1^2 U_1^2} - \rho_1\rho_2 U_1^2 - \rho_1\rho_2 U_2^2 - \cancel{\rho_2^2 U_2^2} - \frac{4g}{k}(\rho_1^2 - \rho_2^2)$$

$$= -\rho_1\rho_2[U_1 - U_2]^2 - \frac{4g}{k}(\rho_1^2 - \rho_2^2)$$

$$\Delta = -\rho_1\rho_2(U_1 - U_2)^2 + \frac{4g}{k}(\rho_2^2 - \rho_1^2)$$

$\Rightarrow$ short wave instability provided $U_1 \neq U_2$

For imag.^{y} part, & instabilities, $\Delta < 0$

$$\Rightarrow -\rho_1\rho_2(U_1 - U_2)^2 + \frac{4g}{k}(\rho_2^2 - \rho_1^2) < 0$$

$$\Rightarrow \frac{4g}{k\rho_1\rho_2}(\rho_2^2 - \rho_1^2) < (U_1 - U_2)^2$$