Quantum Theory Solutions

1. (a) [10 marks] [B]

(i) The stationary state Schrödinger equation (SSSE) for the wave-function $\psi(x)$ of the particle of mass m is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

for $0 \leqslant x \leqslant a$ and $\psi = 0$ elsewhere, where the latter is implied by 'confined', so that the probability density and hence also the wave-function is zero outside the interval. Continuity of ψ therefore gives the boundary conditions as

$$\psi(0) = 0 = \psi(a).$$

(ii) We have

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi,$$

and the boundary conditions force us to consider only circular functions, so that $k^2 := \frac{2mE}{\hbar^2} > 0$. The solution with $\psi(0) = 0$ is $\sin kx$ and the other boundary condition forces $k = n\pi/a$ for positive integer n. The allowed energies are a discrete series

$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},$$

with corresponding wave-functions

$$\psi_n(x) = \alpha_n \sin\left(\frac{n\pi x}{a}\right)$$

for $0 \leqslant x \leqslant a$, and zero elsewhere. For normalisation we want

$$1 = \int_0^a |\psi|^2 dx = |\alpha_n|^2 \frac{a}{2},$$

and without loss of generality we may take $\alpha_n = \sqrt{(2/a)}$. The general solution of the time-dependent SE is a combination of stationary states, so

$$\Psi(x,t) = \sum_1^\infty c_n \psi_n(x) e^{-iE_n t/\hbar},$$

where the coefficients c_n can be fixed, if desired, by $\Psi(x, 0)$.

(b) [15 marks] [B,S]

(i) When the interval has length $2a$, just replace a by $2a$ above, so that, using tildes for this case,

$$\tilde{E}_n = \frac{n^2\pi^2\hbar^2}{8ma^2}, \quad \tilde{\psi}_n = \sqrt{\frac{1}{a}} \sin\left(\frac{n\pi x}{2a}\right).$$

(ii) As noted above, the general solution of SE in the longer interval is

$$\Psi(x,t) = \sum_1^\infty c_n \tilde{\psi}_n(x) e^{-i\tilde{E}_n t/\hbar},$$

inside the interval and zero outside, and we have the initial condition

$$\Psi(x,0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

for $0 \leqslant x \leqslant a$ and zero elsewhere (in particular for $a \leqslant x \leqslant 2a$). To find $\Psi(x,t)$, we first obtain the constants c_n by Fourier series methods (explicit expressions not required).

(iii) Each $|c_n|^2$ is interpreted as the probability of obtaining $\tilde{E}_n$ as the result of measuring the energy. Thus we want c_1. Calculate

$$
\begin{aligned}
c_1 &= \int_0^{2a} \Psi(x,0)\sqrt{\frac{1}{a}}\sin\left(\frac{\pi x}{2a}\right)dx \\
&= \int_0^{a} \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\sqrt{\frac{1}{a}}\sin\left(\frac{\pi x}{2a}\right)dx \\
&= \frac{\sqrt{2}}{a}\int_0^{a} 2\sin^2(\pi x/2a)\cos(\pi x/2a)dx \\
&= \frac{4\sqrt{2}}{3\pi}.
\end{aligned}
$$

Now the required probability is $|c_1|^2 = \frac{32}{9\pi^2}$.

2. (a) [7 marks] [B][S][N]

(i)

$$[A, B] = AB - BA$$

so

$$[AB, C] = ABC - CAB = A(BC - CB) + (AC - CA)B = A[B, C] + [A, C]B,$$

qed

(ii) Induction: true for $n = 1$ and then

$$[X^{n+1}, P] = X[X^n, P] + [X, P]X^n = (n + 1)\mathrm{i}\hbar X^n,$$

qed.

(b) [18 marks] [B,N]

(i) Since ψ is normalised

$$\mathbb{E}_\psi(A) =< \psi, A\psi >,$$

so

$$\mathbb{E}_\psi([H, A]) =< \psi, (HA - AH)\psi >=< \psi, HA\psi > - < \psi, AH\psi >= E < \psi, A\psi > -E < \psi, A\psi >=$$

using reality of E.

(ii) Calculate

$$[H, X] = \frac{1}{2m}[P^2, X] = \frac{1}{2m}(P[P, X] + [P, X]P) = -\frac{\mathrm{i}\hbar}{m}P,$$

so that $\mathbb{E}_\psi(P) = 0$; then

$$[H, P] = \frac{1}{2}k[X^n, P] = \frac{1}{2}\mathrm{i}\hbar k n X^{n-1},$$

so that $\mathbb{E}_\psi(X^{n-1}) = 0$; finally

$$[H, PX] = \frac{1}{2}k[X^n, P]X + \frac{1}{2m}P[P^2, X] = \mathrm{i}\hbar(nV(X) - 2T),$$

so that $\mathbb{E}_\psi(2T - nV) = 0$.
Since $H\psi = (T + V)\psi = E\psi$, at once $\mathbb{E}_\psi(T) + \mathbb{E}_\psi(V) = \mathbb{E}_\psi(T + V) = E$.
Now just solve for $\mathbb{E}_\psi(T)$ and $\mathbb{E}_\psi(V)$.

(iii)

$$\Delta_\psi(A) = \sqrt{\mathbb{E}_\psi((A - \mathbb{E}_\psi(A))^2)},$$

so

$$\Delta_\psi(P) = \sqrt{\mathbb{E}_\psi(P^2)} = \sqrt{2m\mathbb{E}_\psi(T)} = \sqrt{\frac{2mnE}{n+2}}.$$

HUP says

$$\Delta_\psi(P)\Delta_\psi(X) \geqslant \frac{\hbar}{2},$$

and we know $\Delta_\psi(P)$ so we deduce

$$\Delta_\psi(X) \geqslant \frac{\hbar}{2}\sqrt{\frac{n+2}{2mnE}}.$$

3. (a) [4 marks] [B]

Either

$$[L_1, L_2] = \mathrm{i}\hbar L_3 = -[L_2, L_1] \text{ and cyclic permutations,}$$

or

$$[L_i, L_j] = \mathrm{i}\hbar \epsilon_{ijk} L_k,$$

with an explanation of ϵ_{ijk}.

Next

$$(L^2)^* = (L_1^2 + L_2^2 + L_3^2)^* = (L_1^*)^2 + (L_2^*)^2 + (L_3^*)^2 = L^2,$$

and

$$[L^2, L_3] = [L_1^2 + L_2^2 + L_3^2, L_3] = [L_1^2, L_3] + [L_2^2, L_3] + [L_3^2, L_3].$$

Last term vanishes; expand other two:

$$= L_1[L_1, L_3] + [L_1, L_3]L_1 + L_2[L_2, L_3] + [L_2, L_3]L_2,$$

use CRs for L_i:

$$= \mathrm{i}\hbar(-L_1L_2 - L_2L_1 + L_2L_1 + L_1L_2) = 0.$$

Appeal to symmetry for the other cases – qed.

(b) [6 marks] [B]

Calculate

$$L_- = (L_+)^* = (L_1 + \mathrm{i}L_2)^* = L_1 - \mathrm{i}L_2,$$

then

$$[L^2, L_\pm] = [L^2, L_1 \pm \mathrm{i}L_2] = 0 \text{ by previous part.}$$

$$[L_3, L_1 + \mathrm{i}L_2] = [L_3, L_1] + \mathrm{i}[L_3, L_2] = \mathrm{i}\hbar(L_2 - \mathrm{i}L_1) = \hbar L_+$$

and similarly

$$[L_3, L_-] = -\hbar L_-.$$

Keep on:

$$L_+L_- = (L_1 + \mathrm{i}L_2)(L_1 - \mathrm{i}L_2) = (L_1)^2 + (L_2)^2 - \mathrm{i}[L_1, L_2]$$
$$= L^2 - (L_3)^2 + \hbar L_3,$$

while

$$L_-L_+ = L^2 - (L_3)^2 - \hbar L_3.$$

(c) [6 marks] [B]

Calculate

$$L^2\psi_{\pm 1} = L^2 L_\pm \psi_0 = ([L^2, L_\pm] + L_\pm L^2)\psi_0 = L_\pm(2\hbar^2\psi_0) = 2\hbar^2\psi_{\pm 1},$$

so these are eigenvectors of L^2 with the same eigenvalue. The same argument proves the same fact for $\psi_{\pm 2}$.

Then

$$L_3\psi_{\pm 1} = L_3 L_\pm \psi_0 = ([L_3, L_\pm] + L_\pm L_3)\psi_0 = \pm\hbar L_\pm \psi_0 = \pm\hbar\psi_{\pm 1},$$

so these are eigenvectors with eigenvalues respectively $\pm\hbar$. Repeat for $\psi_{\pm 2}$:

$$L_3\psi_{\pm 2} = ([L_3, L_\pm] + L_\pm L_3)\psi_{\pm 1} = (\pm\hbar L_\pm + L_\pm(\pm\hbar))\psi_{\pm 1} = \pm 2\hbar\psi_{\pm 2},$$

and these are eigenvectors with eigenvalues respectively $\pm 2\hbar$.

Since ψ_0 and $\psi_{\pm 1}$ are eigenvectors of L_3 with different eigenvalues they are necessarily orthogonal: $\langle \psi_{\pm 1}, \psi_0 \rangle = 0$.

(d) [9 marks] [B,S,N]

(i)

$$\langle \psi_{+1}, \psi_{+1} \rangle =< L_+\psi_0, L_+\psi_0 >=< \psi_0, L_-L_+\psi_0 >=< \psi_0, (L^2 - (L_3)^2 - \hbar L_3)\psi_0 >$$
$$= 2\hbar^2 ||\psi_0||^2,$$

and similar for ψ_{-1}, but

$$< \psi_{+2}, \psi_{+2} >=< \psi_{+1}, L_-L_+\psi_{+1} >=< \psi_{+1}, (L^2 - (L_3)^2 - \hbar L_3)\psi_{+1} >= 0,$$

and similar for ψ_{-2}. Since the norm is zero, the vector is zero: $\psi_{\pm 2} = 0$.

(ii) Calculate:

$$L_+\psi_{-1} = L_+L_-\psi_0 = (L^2 - (L_3)^2 + \hbar L_3)\psi_0 = 2\hbar^2\psi_0,$$

and the same for $L_-\psi_{+1}$.

L_3 is self-adjoint on V and so can be diagonalised. The norm argument shows the allowed eigenvalues are $0, \pm 1$; anything with $+1$ lowers to one with zero and anything with -1 raises to one with zero; but there is only one dimension of zero-eigenvectors so only 3 dimensions overall.