1 Quantum Theory - Model solutions

1. a.- [B]

The time-dependent Schrödinger equation takes the form

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

[2 marks]

By stationary state we mean a separable solution of this equation, of the form $\Psi(x,t) = \psi(x)T(t)$. Plugging this into the Schrödinger equation it follows $T(t) = e^{-iEt/\hbar}$ and

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V\psi(x) = E\psi(x)$$

[3 marks]

The probability density is given by

$$\rho = |\Psi(x,t)|^2 \tag{1}$$

[1 mark]

and j in one dimension has only one component, and equals

$$j = \frac{i\hbar}{2m}\left(\Psi(x,t)\overline{(\partial_x\Psi(x,t))} - \overline{\Psi(x,t)}\partial_x\Psi(x,t)\right) \tag{2}$$

[1 mark]

For a stationary state $\rho = |\psi(x)|^2$ and $j = 0$ (note that this is consistent with the continuity equation). [2 marks]

b.- [S] The allowed energy levels are $E_n = (n + \frac{1}{2})\hbar\omega$ [1 mark]. The ground state has wave function

$$\psi_0(x) \quad = \quad e^{-\frac{m\omega}{2\hbar}x^2} \tag{3}$$

we can then plug this into the stationary state wave function:

$$-\frac{\hbar^2}{2m}\partial_x^2\psi_0(x) = \left(\frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2x^2\right)\psi_0(x)$$

the first term reproduces the proper energy, the second exactly cancels the potential.[3 marks]

For the second exited state we do the same, but now we need to perform a longer computation, involving

$$-\frac{\hbar^2}{2m}\partial_x^2\frac{m\omega}{\hbar}x^2\psi_0(x) = \left(-\hbar\omega + \frac{5}{2}m\omega^2x^2 - \frac{m^2\omega^3x^4}{2\hbar}\right)\psi_0(x)$$

Now we plug this computation, together with the previous one, into the stationary state equation. We are left with

$$\left(-\hbar\omega + \frac{5}{2}m\omega^2 x^2 - \frac{m^2\omega^3 x^4}{2\hbar}\right)\psi_0(x) + \kappa\left(\frac{\hbar\omega}{2} - \frac{1}{2}m\omega^2 x^2\right)\psi_0(x) + \frac{1}{2}m\omega^2 x^2\left(\frac{m\omega}{\hbar}x^2 + \kappa\right)\psi_0(x) = E\left(\frac{m\omega}{\hbar}x^2 + \kappa\right)\psi_0(x) \quad (4)$$

We now look at different powers of x on both sides. The x^4 power works automatically. The x^2 power gives

$$\frac{5}{2}m\omega^2 - \kappa\frac{1}{2}m\omega^2 + \kappa\frac{1}{2}m\omega^2 = E\frac{m\omega}{\hbar}$$

which results in the correct value for the energy $E = \frac{5}{2}\hbar\omega$. Finally, the piece proportional to x^0 gives

$$-\hbar\omega + \kappa\frac{\hbar\omega}{2} = \kappa E$$

Plugging the correct value for the energy this leads to $\kappa = -1/2$. [4 marks]

c.- [N] First we determine the classical interval. The relevant expectation value is simply $\mathbb{E}_{\Psi_2}(H) = E_2 = \frac{5}{2}\hbar\omega$. This implies

$$\frac{1}{2}m\omega^2 x^2 = \frac{5}{2}\hbar\omega \quad \text{second excited state} \quad (5)$$

[2 marks] In order to compute the probability that the particle is outside the classical region, we need to integrate ρ between x_0 (the positive solution of the above equation) and infinity. We also have to normalise by the same integral between 0 and infinity. We obtain

$$P_1 = \frac{\int_{\sqrt{5}}^{\infty}(\zeta^2+\kappa)^2 e^{-\zeta^2}d\zeta}{\int_0^{\infty}(\zeta^2+\kappa)^2 e^{-\zeta^2}d\zeta} \quad (6)$$

where we have made a change of variables $\zeta^2 = \frac{mw}{\hbar}x^2$.

Integration by parts relates

$$\int_{\zeta_0}^{\infty} e^{-\zeta^2}d\zeta = -\zeta_0 e^{-\zeta_0^2} + 2\int_{\zeta_0}^{\infty}\zeta^2 e^{-\zeta^2}d\zeta \quad (7)$$

and

$$\int_{\zeta_0}^{\infty}\zeta^2 e^{-\zeta^2}d\zeta = -\frac{1}{3}\zeta_0^3 e^{-\zeta_0^2} + \frac{2}{3}\int_{\zeta_0}^{\infty}\zeta^4 e^{-\zeta^2}d\zeta \quad (8)$$

This allows to express all relevant integrals in terms of the given integrals:

$$\int_{\zeta_0}^{\infty}\zeta^2 e^{-\zeta^2}d\zeta = \frac{1}{2}\int_{\zeta_0}^{\infty}e^{-\zeta^2}d\zeta + \frac{1}{2}\zeta_0 e^{-\zeta_0^2} \quad (9)$$

$$\int_{\zeta_0}^{\infty}\zeta^4 e^{-\zeta^2}d\zeta = \frac{3}{4}\int_{\zeta_0}^{\infty}e^{-\zeta^2}d\zeta + \left(\frac{3}{4}\zeta_0 + \frac{1}{2}\zeta_0^3\right)e^{-\zeta_0^2} \quad (10)$$

Taking now $\kappa = -1/2$ into account, we are interested in the combination:

$$\int_{\zeta_0}^{\infty} (\zeta^4 - \zeta^2 + 1/4) e^{-\zeta^2} d\zeta = \frac{1}{2} \int_{\zeta_0}^{\infty} e^{-\zeta^2} d\zeta + (\frac{1}{4}\zeta_0 + \frac{1}{2}\zeta_0^3) e^{-\zeta_0^2}$$

from which we can write:

$$P = 2 \frac{\frac{\sqrt{\pi}}{4} \mathrm{erfc}(\sqrt{5}) + \frac{11\sqrt{5}}{4} e^{-5}}{\frac{\sqrt{\pi}}{4} \mathrm{erfc}(0)}$$

[6 marks]

2.- a. [B] The first part is easy, since P, X are self-adjoint, the only thing we have to do is to change the sign of the i [1mark]. Then by simply using the definitions and expanding we can compute a_+a_- and a_-a_+. For instance:

$$a_+a_- = P^2 - im\omega(PX - XP) + m^2\omega^2 X$$

Using $[X, P] = i\hbar$ and the definition of the Hamiltonian then leads to the result [2marks]. The computation for a_-a_+ is almost identical. Their difference leads to the first commutator[1mark]. The results for the commutators are

$$[a_-, a_+] = 2m\hbar\omega \tag{11}$$

$$[H, a_-] = -\hbar w a_- \tag{12}$$

$$[H, a_+] = \hbar\omega a_+ \tag{13}$$

The second and third can be done by direct computation, but it is simpler to use $H = \frac{1}{2m}a_+a_- + \frac{1}{2}\hbar\omega$. This can then be plugged into the commutators, and using $[a_-, a_+] = 2m\hbar\omega$ we are led to the result. [4marks]

b.-[S] We compute E_0 as follows

$$0 = a_+a_-\psi_0 = 2m(E_0 - \frac{1}{2}\hbar\omega)\psi_0$$

It then follows $E_0 = \frac{1}{2}\hbar\omega$ [2 marks]. Using the commutation relation $[H, a_+] = \hbar\omega a_+$ one can then show inductively that $E_n = (\frac{1}{2} + n)\hbar\omega$ [1 mark]. Next we compute

$$\langle\psi_{n+1}|\psi_{n+1}\rangle = \frac{1}{2m\hbar\omega}\langle\psi_n|a_-a_+|\psi_n\rangle = \frac{1}{\hbar\omega}(E_n + \frac{1}{2}\hbar\omega)\langle\psi_n|\psi_n\rangle = (n+1)\langle\psi_n|\psi_n\rangle \tag{14}$$

[2 marks]

In a normalisation $\langle\psi_0|\psi_0\rangle = 1$ we get $\langle\psi_n|\psi_n\rangle = n!$ [1mark], while $\langle n|n'\rangle = 0$ for $n \neq n'$ [1mark] (this can be shown, for instance, by acting with H on both sides).

c.-[N] Let us consider the following

$$||\frac{1}{\sqrt{2m\hbar\omega}}a_-\psi_{n+1}||^2 = \frac{1}{2m\hbar\omega}\langle\psi_{n+1}|a_+a_-|\psi_{n+1}\rangle = (n+1)\langle\psi_{n+1}|\psi_{n+1}\rangle = (n+1)^2\langle\psi_n|\psi_n\rangle$$

It then follows

$$\frac{1}{\sqrt{2m\hbar\omega}}a_-\psi_{n+1} = (n+1)\psi_n$$

Now propose the following expansion

$$\Psi_\alpha = \sum_n c_n\psi_n$$

and act on both sides with $\frac{1}{\sqrt{2m\hbar\omega}}a_-$. We obtain

$$\alpha \sum_n c_n \psi_n = \sum_n c_n n \psi_{n-1} = \sum_n c_{n+1}(n+1)\psi_n$$

which gives a recursion relation among the coefficients, and fixes the coefficients up to a constant:

$$c_{n+1}(n+1) = \alpha c_n \to c_n = \frac{\alpha^n}{n!}$$

[4 marks]. Now we can compute the norm

$$||\Psi_\alpha||^2 = \sum_n \frac{\alpha^{2n}}{n!} = e^{\alpha^2}$$

so that we divide by the square root of this if we want to normalise. [1mark]

The given observable has eigenvalues 1 and -1, so that these are the possible values. [1mark]

The probability of measuring $+1$ is

$$P(1) = \frac{1}{||\Psi_\alpha||^2} \sum_{n\ even} \frac{\alpha^{2n}}{n!} = \frac{1}{2}(1 + e^{-2\alpha^2})$$

while

$$P(-1) = \frac{1}{||\Psi_\alpha||^2} \sum_{n\ odd} \frac{\alpha^{2n}}{n!} = \frac{1}{2}(1 - e^{-2\alpha^2})$$

[3marks]

3.- a [B] The commutation relations can be written as

$$[J_i, J_j] = i\hbar \sum_k \epsilon_{ijk} J_k$$

[1mark]

Self-adjointness of J^2 follows from $(J_i J_i)^* = J_i^* J_i^* = J_i J_i$ [1mark]. In order to show J^2 commutes with each of them, we can choose, let's say J_1, then:

$$[J^2, J_1] = [J_2^2, J_1] + [J_3^2, J_1] = J_2[J_2, J_1] + [J_2, J_1]J_2 + J_3[J_3, J_1] + [J_3, J_1]J_3$$

But then $[J_1, J_2] = i\hbar J_3 \to [J_2, J_1] = -i\hbar J_3$ and $[J_3, J_1] = i\hbar J_2$. Plugging this above we see that the four commutators exactly cancel out (it is also possible to use an index notation, as done in the notes). [3marks]

b.-[B] $[J^2, J_\pm] = 0$, since we have just proven that J^2 commutes with both J_1 and J_2 [1mark]. Furthermore:

$$[J_3, J_\pm] = [J_3, J_1] \pm i[J_3, J_2] = i\hbar J_2 \mp i(i\hbar J_1) = \pm\hbar J_\pm$$

[2 marks]. For the next part:

$$J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2) = J_1^2 + J_2^2 - i[J_1, J_2] = J^2 - J_3^2 + \hbar J_3$$

$$J_- J_+ = (J_1 - iJ_2)(J_1 + iJ_2) = J_1^2 + J_2^2 + i[J_1, J_2] = J^2 - J_3^2 - \hbar J_3$$

[2 marks]

c.-[S] For the norms, we simply compute:

$$||\psi_+||^2 = \langle\psi_0|J_- J_+ \psi_0\rangle = \langle\psi_0|(J^2 - J_3^2 - \hbar J_3)\psi_0\rangle = 2\hbar^2$$

and for $||\psi_-||^2$ we obtain exactly the same result [2 marks]. Now we consider

$$\chi_0 = \frac{1}{2\hbar}(\psi_+ - \psi_-)$$

Now we act on this with $J_1 = \frac{1}{2}(J_+ + J_-)$ and note that

$$J_+\psi_+ = J_-\psi_- = 0 \tag{15}$$

$$J_+\psi_- = J_+ J_- \psi_0 = 2\hbar^2\psi_0 \tag{16}$$

$$J_-\psi_+ = J_- J_+ \psi_0 = 2\hbar^2\psi_0 \tag{17}$$

It then follows that χ_0 is an eigenvector of J_1 with eigenvalue equal zero. [2 marks]. For the next part we proceed in a similar way. We assume

$$\chi_+ = \alpha_+\psi_+ + \alpha_0\psi_0 + \alpha_-\psi_-$$

And we act with $J_1 = \frac{1}{2}(J_+ + J_-)$ on both sides of this equation. The L.H.S. is $\hbar\chi_+$ by assumption. And for the R.H.S. we use the relations found above. Equating both sides then leads to

$$\hbar(\alpha_+\psi_+ + \alpha_0\psi_0 + \alpha_-\psi_-) = (\hbar^2\alpha_+ + \hbar^2\alpha_-)\psi_0 + \frac{1}{2}\alpha_0(\psi_+ + \psi_-)$$

equating this coefficients in front of $\psi_0, \psi_\pm$ leads to

$$\chi_+ = \alpha(\psi_+ + 2\hbar\psi_0 + \psi_-)$$

where α is a constant to be fixed. Computing the norm of this we obtain:

$$||\chi||^2 = |\alpha|^2(8\hbar^2)$$

so that $\alpha = \frac{1}{\sqrt{8}\hbar}$. The computations for χ_- are identical and we obtain

$$\chi_+ = \frac{1}{2\sqrt{2}\hbar}(\psi_+ + 2\hbar\psi_0 + \psi_-) \tag{18}$$

$$\chi_- = \frac{1}{2\sqrt{2}\hbar}(\psi_+ - 2\hbar\psi_0 + \psi_-) \tag{19}$$

[4marks]

d.-[N] If we measure J_3 on χ_+ we can obtain $\pm\hbar$ and 0. In order to compute the probabilities we have to be careful, because $\psi_\pm$ are not normalised (their norm square is $2\hbar^2$ as seen above).

The coefficient in front of the normalised ψ_0 is $1/\sqrt{2}$, hence the probability to measure zero is 1/2. The probability in front for $\frac{1}{\sqrt{2}\hbar}\psi_+$ (the normalised eigenvector) is 1/2, so that the probability of measuring $\hbar$ is 1/4, and the same for the probability of measuring $-\hbar$. [3marks]

Now we measure J_3 in χ_+ and obtain zero. The wave function after measurement is then

$$\chi_+ \to \frac{1}{\sqrt{2}}\psi_0$$

Now we want to measure J_1, hence we need to re-express this in terms of J_1 eigenvectors. We obtain

$$\frac{1}{\sqrt{2}}\psi_0 = \frac{1}{2}(\chi_+ - \chi_-)$$

Now, for a probabilistic interpretation we need to normalise the states, such that the total of the probabilities adds up to one, of course. In any case, we see that the values that can be measure are $\hbar$ and $-\hbar$ with an equal probability of 1/2. [4marks]