A11473W1

**DEGREE OF MASTER OF SCIENCE**

**Mathematical Modelling and Scientific Computing**

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# Mathematical Methods I

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**Hilary Term 2023**

**Thursday, 12 January 2023, 9:30am to 12.00pm**

***This exam paper contains two sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best two answers, making a total of four answers.***

***Candidates may bring a summary sheet into this exam consisting of (both sides of) one sheet of A4 paper containing material prepared in advance in accordance with the guidance given by the Mathematical Institute.***

***Please start the answer to each question in a new booklet.***

**Do not turn this page until you are told that you may do so**

# Applied Partial Differential Equations

1. Consider solutions $u(x,t)$ of the partial differential equation

$$u_t - u_{xx} = 0 \qquad \text{on } 0 < x < \infty, \tag{1}$$

that satisfy the initial condition

$$u(x,0) = 0 \qquad \text{for } x > 0, \tag{2}$$

and the boundary conditions

$$u(0,t) = g(t) \qquad \text{for } t > 0, \tag{3}$$

$$\lim_{x\to\infty} u(x,t) = 0 \qquad \text{for } t > 0, \tag{4}$$

for some prescribed function $g(t)$.

(a) [6 marks] For the case $g(t) = 1$, show that, for a suitable choice of $a$, $b$ and $c$, the problem (1)–(4) is invariant under the scalings

$$t = \varepsilon^a \bar{t}, \quad x = \varepsilon^b \bar{x}, \quad u(x,t) = \varepsilon^c \bar{u}(\bar{x}, \bar{t}), \tag{5}$$

for all $\varepsilon > 0$.

(b) [6 marks] What is the most general form of $g(t)$, bounded as $t \to 0$, for which a suitable choice of $a$, $b$ and $c$ with $a \neq 0$ is possible, such that (1)–(4) are invariant under the scalings (5)? Give reasons for your answer and determine $a$, $b$ and $c$ in this case.

(c) [13 marks] For the case $g(t) = t$, determine constants $\alpha$ and $\beta$ such that

$$u(x,t) = t^\alpha f(\xi) \quad \text{with } \xi = x/t^\beta,$$

is a self-similar solution of (1)–(4). State the resulting boundary value problem for a second order ordinary differential equation for $f$. Determine the self-similar solution, $u$, explicitly, using the function

$$H(\xi) = \int_\xi^\infty \frac{\exp(-\frac{1}{4}s^2)}{(s^2+2)^2}\,\mathrm{d}s.$$

[*To solve the ordinary differential equation for $f$, use the substitution $f(\xi) = (\xi^2+2)p(\xi)$ and solve the resulting differential equation for $p(\xi)$. You may also use, without proof, that $H(0) = \sqrt{\pi}/8$.*]

2. Consider the first order quasilinear partial differential equation

$$u_t + \frac{1}{2}(u^2)_x = 0, \qquad -\infty < x < \infty, \tag{1}$$

with initial data

$$u(x,0) = g(x), \qquad -\infty < x < \infty, \tag{2}$$

where

$$g(x) \equiv \begin{cases} 0, & \text{if } |x| > 1/2, \\ x, & \text{if } |x| \leqslant 1/2. \end{cases} \tag{3}$$

(a) [4 marks] Determine the speed for a shock with left and right states $u = u_-$ and $u = u_+$, respectively. Specify a condition on $u_-$ and $u_+$ for which the shock is causal.

(b) [12 marks] Show that the shocks present in the initial data (2) are causal. Hence use the method of characteristics to determine the solution of the initial value problem (1)–(3), for all $t \geqslant 0$. You should also determine the trajectory of the shocks.

(c) [9 marks] Now consider the initial condition

$$u(x,0) = \sum_{k=-\infty}^{\infty} g(x - 2k),$$

for (1), where $g$ is as defined in (3).

(i) Sketch the initial condition.

(ii) Determine the solution for $0 \leqslant t < 3$. Explain what happens at $t = 3$.

(iii) Continue the solution for $t \geqslant 3$. What is the limit of $u(x,t)$ as $t \to \infty$?

3. Consider the partial differential equation

$$2(u_y - u_x)^3 + (x + y)u_x + (x + y)u_y + 2u = 0, \tag{1}$$

for $u(x, y)$.

(a) [10 marks] Formulate Charpit's equations for (1) for $x$, $y$, $u$, $p = u_x$ and $q = u_y$. Find the parametric solutions for $p$, $q$, $x$, $y$ for initial data given by the smooth functions $p_0(s)$, $q_0(s)$, $x_0(s)$ and $y_0(s)$.

[*You may find it helpful to add and subtract suitably chosen pairs of Charpit's equations. You are not required to find the parametric solution for* $u$.]

(b) [8 marks] For the initial condition of (1) given by

$$u(s, s) = 2s^3 \quad \text{for} \quad 0 \leqslant s \leqslant 1, \tag{2}$$

determine the appropriate initial data for Charpit's equations. Hence obtain the characteristics in the $(x, y)$ plane for the solution of (1) and (2) in parametric form.

(c) [7 marks] Indicate on a sketch, and describe carefully, where the solution of (1) is uniquely determined in the half-plane $y \geqslant x$ by the initial data specified in (2).

[*You are not required to determine* $u(x, y)$.]

4. (a) [15 marks] Consider Laplace's equation on the domain $\mathcal{D} = \{(x, y, z) : x > 0,\ y > 0,\ z > 0\}$ in $\mathbb{R}^3$, with boundary conditions $u = g_1(y, z)$ on $x = 0$, $u_y = g_2(x, z)$ on $y = 0$ and $u_z = g_3(x, y)$ on $z = 0$, where $g_1$, $g_2$ and $g_3$ are prescribed functions. State the boundary-value problem satisfied by the corresponding Green's function $G(x, y, z; \xi, \eta, \zeta)$ on $\mathcal{D}$, and hence obtain $G$ explicitly.

[*You may assume without proof that*

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}\ ,$$

*is the Green's function for the Laplace operator in* $\mathbb{R}^3$.]

(b) [10 marks] Consider the diffusion equation

$$u_t - u_{xx} - u_{yy} = f(x, y, t) \quad \text{in} \quad \mathbb{R}^2, \tag{1}$$

with initial condition

$$u(x, y, 0) = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{2}$$

and boundary condition

$$u(x, y, t) \to 0 \qquad \text{as } x^2 + y^2 \to \infty. \tag{3}$$

The Green's function $G(x, y, t; \xi, \eta, \tau)$ for this problem is defined by

$$\begin{aligned} G_t + G_{xx} + G_{yy} &= 0, & t < \tau,\ (x, y) \in \mathbb{R}^2, \\ G &= \delta(x-\xi)\delta(y-\eta), & t = \tau,\ (x, y) \in \mathbb{R}^2, \\ G &\to 0, & t < \tau,\ x^2 + y^2 \to \infty, \end{aligned}$$

where $\delta$ is the Dirac delta function.

By integrating $G(u_t - \nabla^2 u) - u(-G_t - \nabla^2 G)$ over $0 < t < \tau$ and the disc $x^2 + y^2 < R^2$ of radius $R$, show that in the limit $R \to \infty$, you obtain the formula

$$u(\xi, \eta, \tau) = \int_0^\tau \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, t; \xi, \eta, \tau) f(x, y, t)\ \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t,$$

for the solution of (1)–(3). Here, $\nabla^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)$ denotes the two-dimensional Laplace operator with respect to $x$ and $y$.

## Supplementary Applied Mathematics

5. (a) [6 marks] Determine the general solution of the differential equation given by

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x \frac{\mathrm{d}y}{\mathrm{d}x} + (1+\theta)y = 0, \tag{1}$$

where $\theta$ is a positive constant.

(b) [9 marks] Using the results from part (a) or otherwise, determine the eigenvalues $\lambda_k$ and eigenfunctions $y_k(x)$ of the following boundary-value problem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{\lambda}{x^3} y = 0, \quad \text{on } 1 < x < e, \tag{2}$$

$$y(1) = 0, \; y(e) = 0. \tag{3}$$

(c) [10 marks] Suppose now that $y(x)$ satisfies:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 1, \quad \text{on } 1 < x < e, \tag{4}$$

$$y(1) = 0, \; y(e) = 0. \tag{5}$$

Obtain a solution to (4)–(5) of the form

$$y(x) = \sum_{k=1}^{\infty} c_k y_k(x), \tag{6}$$

where $y_k(x)$ is an eigenfunction of (2)–(3). Determine analytic expressions for the coefficients $c_k$.

6. You are given that $y(x)$ solves the following boundary-value problem:

$$L[y] \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = f(x), \quad \text{on } a < x < b, \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(a) = \alpha \quad \text{and} \quad y(b) = \beta. \tag{2}$$

In (1)–(2), $p(x)$, $q(x)$ and $f(x)$ are real functions, with $p(x) \geqslant 0$ for $a \leqslant x \leqslant b$, and $\alpha$ and $\beta$ are constant parameters.

(a) [10 marks] The Green's function $G(x, \xi)$ satisfies

$$L[G] = \delta(x - \xi). \tag{3}$$

where $L[y]$ is defined in (1) and $\delta(x)$ is the Dirac delta function.
By considering $GL[y] - yL[G]$, derive an expression for $y(\xi)$ in terms of the Green's function and the functions $p(x)$ and $f(x)$. Explain carefully what boundary and continuity conditions $G(x, \xi)$ should satisfy.

(b) [15 marks] Consider now the following boundary value problem for $y(x)$:

$$L[y] \equiv \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 1, \quad \text{on } 0 < x < 1, \tag{4}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) = 0 \quad \text{and} \quad y(1) = \beta. \tag{5}$$

(i) Determine the Green's function $G(x, \xi)$ associated with (4)–(5).
(ii) Using the results from part (a), construct an explicit analytical solution for $y(\xi)$.