DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A1 Mathematical Methods I

HILARY TERM 2019

THURSDAY, 16 JANUARY 2020, 9.30am to 12.00pm

This exam paper contains two sections. You may attempt as many questions as you like but you must answer at least one question in each section. Your best answer in each section will count, along with your next best two answers, making a total of four answers.

Please start the answer to each question in a new booklet.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Applied Partial Differential Equations

1. (a) [12 marks] Assume that the evolution of a blob of fluid in a porous medium $0 < z < h(r,t)$ is given in cylindrical coordinates by the nonlinear PDE

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}h^2\right), \tag{1a}$$

with boundary conditions

$$\frac{\partial h}{\partial r}(0,t) = 0 \quad \text{and} \quad h(a(t),t) = 0, \tag{1b}$$

where $a(t)$ is the lateral extent of the blob.
Using the ansatz

$$h(r,t) = t^{\alpha}H(\eta), \qquad r = t^{\beta}\eta, \qquad a(t) = t^{\beta}\zeta.$$

for a self-similar solution, state all the conditions on the exponents α and β you can obtain from the partial differential equation and the boundary conditions, and state the resulting boundary value problem for H.
[You are not required to solve the bounday value problem.]

(b) [13 marks] Show that

$$I = \int_0^{a(t)} rh(r,t)\mathrm{d}r,$$

is constant along any solution of (1). Use this to obtain another relation between α and β, and find their values explicitly. State and solve the resulting boundary value problem for H and ζ so that $I = 1$.

2. Consider the first order partial differential equation for $u(x, y)$:

$$F(x, y, u, p, q) = 0,$$

where $p = \frac{\partial u}{\partial x}$, $q = \frac{\partial u}{\partial y}$ and F is C^2 in its arguments. Suppose that the initial data is specified on a curve in the (x, y)-plane so that

$$x = x_0(s), \quad y = y_0(s), \quad u = u_0(s), \quad \text{for } s_1 \leqslant s \leqslant s_2.$$

(a) [8 marks] State Charpit's equations for this problem, together with appropriate initial data for their solution. Show that $F = 0$ along their solution.

(b) [10 marks] Find a solution of Charpit's equations in parametric form for

$$p - \frac{1}{4}q^2 - y^2 = 0, \tag{2}$$

with initial data $x = 0$, $y = s$, $u = u_0(s)$.
[You might find the following identities helpful: $\cos^2 \tau = \frac{1+\cos(2\tau)}{2}$, $\sin^2 \tau = \frac{1-\cos(2\tau)}{2}$, $\sin(2\tau) = 2 \sin \tau \cos \tau$.*]*

(c) [7 marks] Determine all solutions $u(x, y)$ to (2) for the initial data

$$u(0, y) = 0 \quad \text{for } -1 \leqslant y \leqslant 1.$$

Determine the domain in the (x, y)-plane where your solutions are defined and sketch it. Specify all points where rays intersect.

3. Consider the first order quasilinear partial differential equation

$$u_t + \frac{1}{2}(u^2)_x = 0. \tag{3}$$

(a) (i) [5 marks] State the characteristic equations for (3) with initial data corresponding to $u(x,0) = u_0(x)$, $a < x < b$, for continuous u_0, and obtain the solution for this system in parametric form.

(ii) [5 marks] State conditions for the speed of a shock for (3) and its causality in terms of its left and right state u_- and u_+, respectively.

(b) Now consider (3) with initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0, \\ -1 & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x. \end{cases} \tag{4}$$

(i) [7 marks] Determine the explicit solution for $u(x,t)$ for $t < t_1 = 1$. Why does this form of the solution cease to be valid at $t = 1$?

[Hint: For non-causal discontinuities in the initial data, say from u_- to u_+ at $x = x_c$, seek solutions of the characteristic equations with initial values $x(0) = x_c$, $u(0) = \xi$, with the parameter ξ in the range between the two values $u_\pm$.]

(ii) [6 marks] Continue the solution to $t > 1$, giving the explicit form for $u(x,t)$ and the trajectory of any shocks.

(iii) [2 marks] Which other transition occurs and at which time $t_2 > 1$? What is the solution for $t > t_2$?

4. (a) (i) [5 marks] Give the general classification for the system of PDEs

$$Au_x + Bu_y = c$$

for the vector of unknowns $u = u(x, y) \in \mathbb{R}^2$, where $A = A(u, x, y)$, $B = B(u, x, y)$ are given real 2×2 matrices and $c(u, x, y)$ is a given function with values in $\mathbb{R}^2$, stating the conditions for the system to be hyperbolic, elliptic or parabolic.
Give the definition of a characteristic curve $y = y(x)$ for this system.

(ii) [5 marks] Classify the shallow water equation

$$h_t + q_x = 0, \tag{5a}$$

$$q_t + \left(\frac{q^2}{h} + \frac{h^2}{2}\right)_x = 0, \tag{5b}$$

with $q > 0$, $h > 0$, giving explicit formulae for the characteristic slopes λ in terms of q and h.

(iii) [5 marks] State the Rankine-Hugoniot condition for the speed s of shocks in solutions of (5) with left and right states (h_-, q_-) and (h_+, q_+) respectively and derive an explicit expression for s in the case of a weak shock i.e. where both $|h_+ - h_-| \ll 1$ and $|q_+ - q_-| \ll 1$.
[Hint: For the last part, let $h_- = h_0$, $h_+ = h_0 + \delta\phi$, $q_- = q_0$, $q_+ = q_0 + \delta\psi$, with $0 < \delta \ll 1$, and constants $h_0 > 0$ and $q_0 > 0$, and determine, to order $O(\delta)$, a linear system of equations for ϕ and ψ.]

(b) [10 marks] For

$$\begin{aligned} u_t - u_{xx} &= f(x,t), & t > 0,&\quad 0 < x < \infty, \\ u &= u_0(x) & t = 0,&\quad 0 < x < \infty, \\ u_x &= g(t), & t > 0,&\quad x = 0, \\ u &\to 0, & t > 0,&\quad x \to \infty, \end{aligned} \tag{6}$$

state the problem which the Green's function $G(x, t)$ has to satisfy and solve it to find $G(x, t)$ explicitly. State the general solution of (6) using G.
[You may use, without proof, that

$$K(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

satisfies $K_t - K_{xx} = 0$ for $-\infty < x < \infty$, $t > 0$ and $K(x, 0) = \delta(x)$, where $\delta(x)$ denotes the Dirac delta-function.]

Section B: Supplementary Mathematical Methods

5. (a) [7 marks] Find the general solution of the differential equation,

$$Ly(x) \equiv \frac{d^2y}{dx^2}(x) - \alpha\frac{dy}{dx}(x) - 2y(x) = \lambda y(x), \tag{7}$$

where $\alpha, \lambda \in \mathbb{R}$ are constants. Determine all forms that real solutions take as α and λ vary, and note ranges of these parameters where each solution is valid.

(b) Consider the boundary value problem,

$$Ly(x) = f(x), \quad y(-1) = 0, \quad y(1) + \frac{dy}{dx}(1) = 0, \tag{8}$$

with the operator L as defined above.

(i) [9 marks] Compute the corresponding eigenvalues and eigenfunctions of this boundary value problem. Note carefully which of the solutions found from part (a) apply in the eigenvalue problem as α varies, noting any special values of α.
[*For some eigenvalues, you can write a transcendental equation which determines the eigenvalues, rather than explicitly computing them.*]

(ii) [5 marks] Determine the adjoint eigenfunctions, and hence the solution y to (8) carefully checking any necessary conditions on α and λ. Compute all integrals explicitly (except those involving $f(x)$).

(iii) [4 marks] Find a value of α for which the eigenvalues and eigenfunctions can be explicitly computed and state them. Given the eigenfunctions y_n and adjoint eigenfunctions w_m, determine the value of $< y_n, w_m >$ for all $n, m \geqslant 0$ for this specific value of α.

6. Consider the boundary value problem

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x), \qquad y(0)=0, \quad y(1)=0. \tag{9}$$

(a) [6 marks] Find $p(x)$, $q(x)$ and $F(x)$ so that the problem (9) can be written in the self-adjoint form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x)y = F(x), \qquad y(0)=0, \quad y(1)=0. \tag{10}$$

(b) [6 marks] Find two linearly independent solutions to the homogeneous version of (9) (or equivalently (10)).

[*Hint: Try standard functions for ODEs of different types.*]

(c) [8 marks] Find the Green's function for (10). Use this to write the general solution to this problem.

(d) [5 marks] Find the explicit solution to the problem

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x(x-1)^2, \qquad y(0)=0, \quad y(1)=2.$$